Shimizu's Lemma for Complex Hyperbolic Space and its Application

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1. Shimizu's Lemma for Real Hyperbolic space

2. Preliminaries

3. Shimizu's Lemma for Complex Hyperbolic Space

4. Complex Hyperbolic Triangle Groups of type (n, n, ∞)

Shimizu's Lemma

$G < PSL(2,\mathbb{C})$ discrete

$G \ni A(z) = z + t \quad (t > 0)$

parabolic

$$G \ni B(z) = \frac{az+b}{cz+d} \quad (ad - bc = 1,\ c \neq 0)$$

$$I(B) = \{ z \in \hat{\mathbb{C}} \mid | z - \underbrace{(-\tfrac{d}{c})}_{= B^{-1}(\infty)} | = \underbrace{\tfrac{1}{|c|}}_{= r_B} \}$$

isometric circle

$$\Longrightarrow \quad r_B \leqq t$$

$$\left(= |AB(\infty) - B(\infty)|^{\frac{1}{2}}\ |AB^{-1}(\infty) - B^{-1}(\infty)|^{\frac{1}{2}} \right)$$

$$(1 \leqq |c| t)$$

Jørgensen's inequality

$G < PSL(2, \mathbb{C})$: discrete
(non-elementary)

$A, B \in G$

$$\Longrightarrow |\mathrm{tr}^2(A) - 4| + |\mathrm{tr}[A, B] - 2| \geqq 1$$

Jørgensen's $\Longrightarrow$ Shimizu's lemma

Corollary

G : Fuchsian group
acting on the upper half plane

$$G(z) \ni A(z) = z + t \qquad (t > 0)$$

parabolic

$\Longrightarrow$ $U = \{ z \mid \mathrm{Im}(z) > t \}$ is precisely invariant under G_∞ (in G)

$$\left(\begin{array}{ll} \text{For } B \in G_\infty, & B(U) = U \\ \text{For } B \in G \setminus G_\infty, & B(U) \cap U = \phi \end{array} \right)$$

$z = x + it$

$$|\mathrm{Im}\, B(z)| = |\mathrm{Im}\, B(x+it)| = \frac{t}{(cx+d)^2 + c^2t^2} \leq \frac{t}{c^2t^2} \leq t$$

Shimizu's Lemma
$(ct \geq 1)$

$V \, (= \mathbb{C}^{2,1})$: 3-dim. complex vector space

$$\langle Z, W \rangle = Z_1 \overline{W_1} + Z_2 \overline{W_2} - Z_3 \overline{W_3}$$

for $Z = \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix}, \; W = \begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} \in V$

$U(1, 2; \mathbb{C})$

$= \{$ automorphism A of V preserving $\langle \, , \, \rangle \}$

$$\langle AZ, AW \rangle = \langle Z, W \rangle$$

$$V_- = \{ Z \in V \mid \langle Z, Z \rangle < 0 \}$$

$$V_0 = \{ Z \in V \mid \langle Z, Z \rangle = 0 \}$$

$$\pi : V_- \cup V_0 \setminus \{0\} \longrightarrow \mathbb{C}^2$$

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \longmapsto \begin{pmatrix} z_1/z_3 \\ z_2/z_3 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

For $z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in V_- \ (V_0)$

$$\langle z, z \rangle = |z_1|^2 + |z_2|^2 - |z_3|^2 \underset{(=)}{<} 0$$

$$\left|\frac{z_1}{z_3}\right|^2 + \left|\frac{z_2}{z_3}\right|^2 \underset{(=)}{<} 1$$

$$|z_1|^2 + |z_2|^2 \underset{(=)}{<} 1$$

$$\pi(V_-) = H^2_{\mathbb{C}} = \{ (z_1, z_2) \in \mathbb{C}^2 \,/\, |z_1|^2 + |z_2|^2 < 1 \}$$

complex unit ball

$$\left(= \{ (z_1, z_2) \in \mathbb{C}^2 \mid \mathrm{Im}(z_1) > \tfrac{1}{2} |z_2|^2 \} \right)$$

Siegel domain

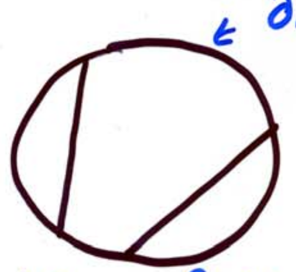
d : Bergman metric

$(H^2_{\mathbb{C}}, d)$: complex hyperbolic space

- holomorphic sectional curvature $\equiv -1$

- totally geodesic subspaces in $H^2_{\mathbb{C}}$

1) $H^1_{\mathbb{R}}$



Klein-Beltrami model

2) $H^2_{\mathbb{R}}$ $(= H^2_{\mathbb{C}} \cap \mathbb{R}^2)$

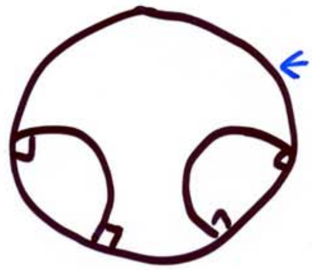
$K = -\frac{1}{4}$

real slice

3) $H^1_{\mathbb{C}}$ $(= H^2_{\mathbb{C}} \cap \mathbb{C})$

$K = -1$

complex slice
(complex geodesic)



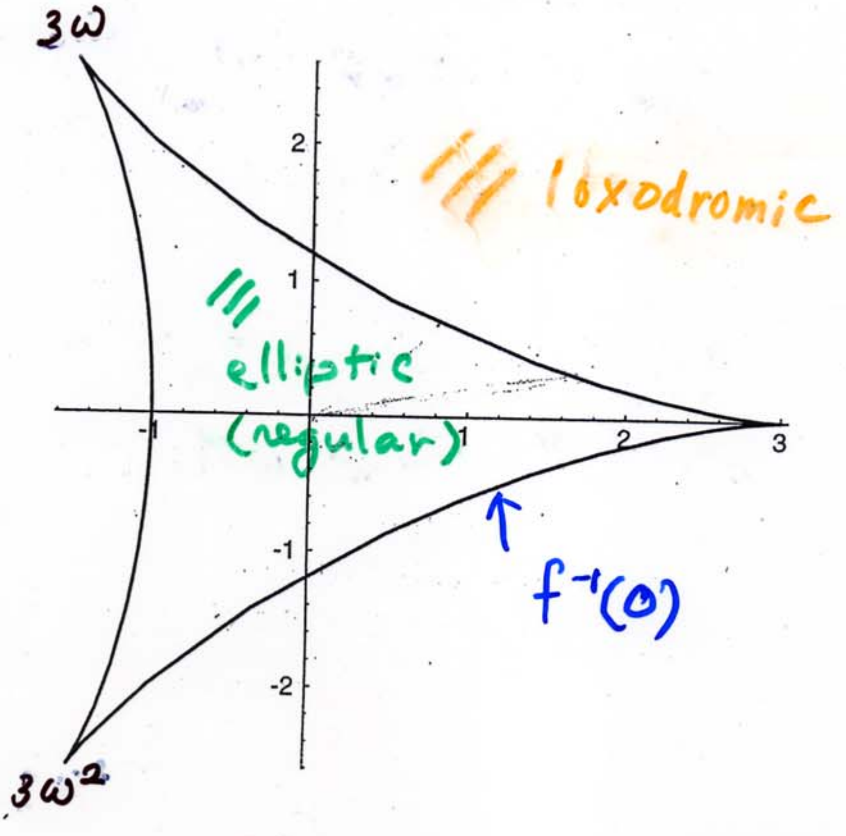
Poincaré model

$$\boxed{\begin{aligned} PU(1,2;\mathbb{C}) &= U(1,2;\mathbb{C}) / (\text{center}) \\ &= \{\text{all biholomorphic isometries of } H^2_{\mathbb{C}}\} \end{aligned}}$$

Classification of elements of $PU(1,2;\mathbb{C})$

$A \ (\neq id) \in PU(1,2;\mathbb{C})$

A
- elliptic
  - regular elliptic
  - boundary elliptic
- loxodromic
- parabolic
  - unipotent
    - vertical translation
    - non-vertical Heisenberg translation
  - non-unipotent (screw parabolic)



$$f(\tau) = |\tau|^4 - 8\,\mathrm{Re}(\tau^3) + 18|\tau|^2 - 27$$

$$\tau = \mathrm{trace}(A)$$

H-coordinates system

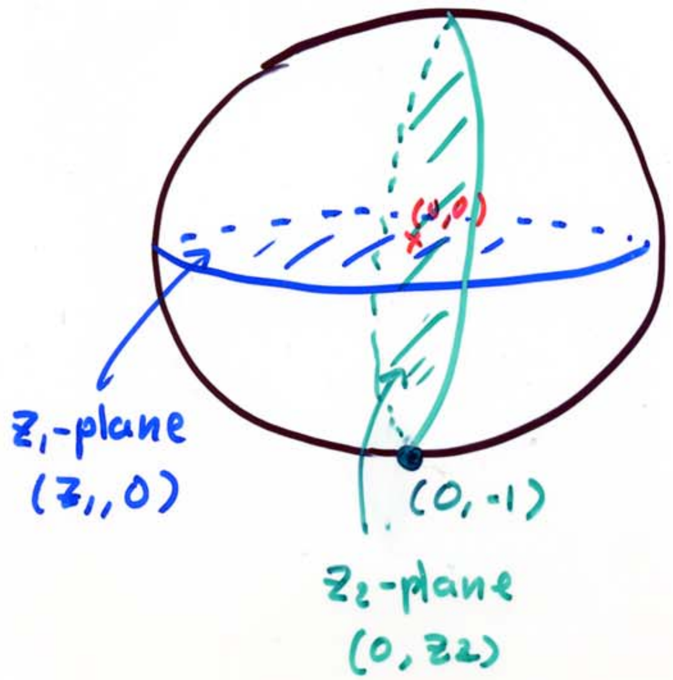
$$\overline{H^2_{\mathbb{C}}} \setminus \{\infty\} \longleftrightarrow \mathbb{C} \times \mathbb{R} \times \mathbb{R}_{(\geq 0)}$$

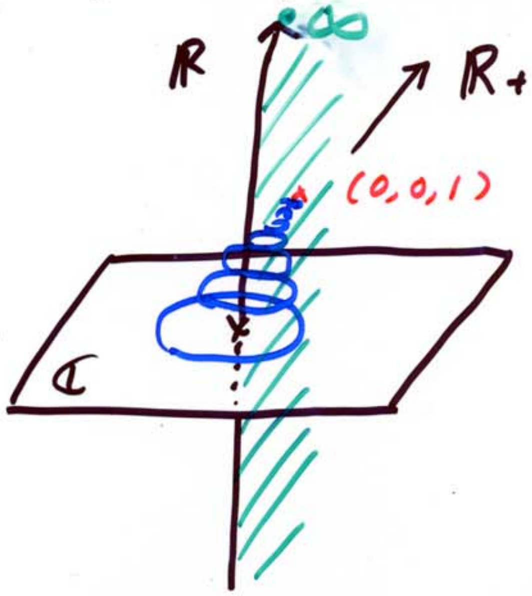
$$\begin{array}{ccc} \Psi & & \Psi \\ (z_1, z_2) & \longleftrightarrow & (\zeta, v, u)_H \end{array}$$

$$\begin{cases} \zeta = \dfrac{z_1}{1+z_2} \\ v = -\operatorname{Im}\left(\dfrac{1-z_2}{1+z_2}\right) \\ u = \operatorname{Re}\left(\dfrac{1-z_2}{1+z_2}\right) - \left|\dfrac{z_1}{1+z_2}\right|^2 \end{cases}$$

$$(0, -1) \longleftrightarrow \infty$$

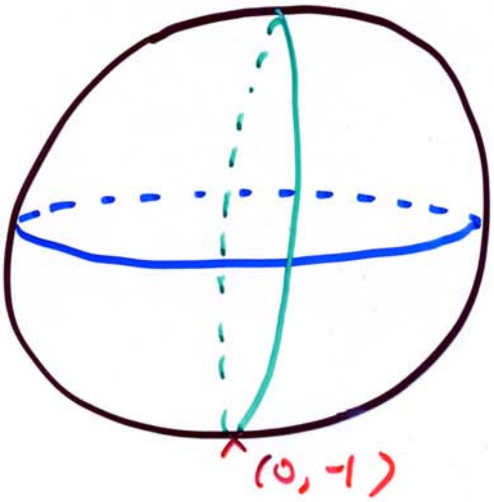
$$H^2_{\mathbb{C}} = \{ (\zeta, v, u)_H \mid \zeta \in \mathbb{C},\ v \in \mathbb{R},\ u \in \mathbb{R}_+ \}$$



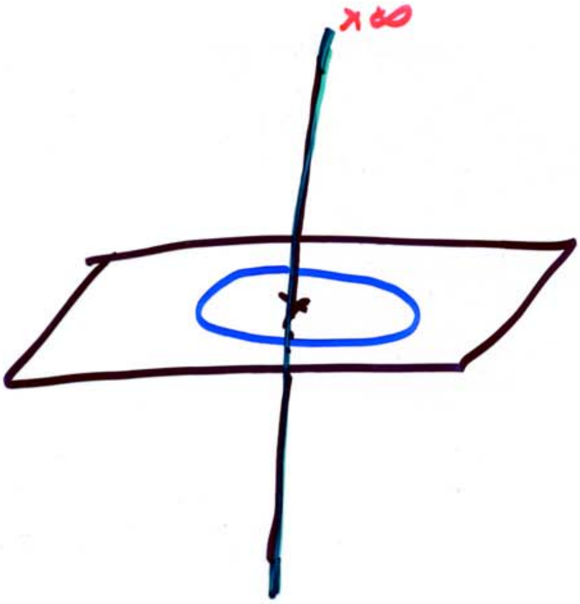


Complex slices

Chains



$\partial H^2_{\mathbb{C}}$ $(= S^3)$



$\mathcal{H} = \mathbb{C} \times \mathbb{R}$

Heisenberg space

$$\begin{array}{ccc} \partial H^2_{\mathbb{C}} & \longrightarrow & \mathcal{H} \\ \rotatebox{90}{$\in$} & & \\ (z_1, z_2) & \longmapsto & \left(\dfrac{z_1}{1+z_2}, \, -\mathrm{Im}\left(\dfrac{1-z_2}{1+z_2} \right) \right) \\ (0, -1) & \longmapsto & \infty \end{array}$$

Cygan metric

For $p = (\zeta_1, v_1, u_1)_H,\ q = (\zeta_2, v_2, u_2)_H \in \overline{H^2_{\mathbb{C}}} \setminus \{\infty\}$

$$\rho(p, q) = \left| |\zeta_1 - \zeta_2|^2 + |u_1 - u_2| + i v_1 - i v_2 + 2i\,\mathrm{Im}(\zeta_1 \bar{\zeta}_2) \right|^{\frac{1}{2}}$$

- Cygan metric is invariant under Heisenberg translations

- $\rho|_{\partial H^2_{\mathbb{C}} \setminus \{\infty\}}$: Heisenberg metric

- For $p = (p_1, p_2),\ q = (q_1, q_2) \in \partial H^2_{\mathbb{C}} \setminus \{\infty\}$

$$\rho(p, q) = |\langle \mathbf{P}, \mathbf{Q} \rangle|^{\frac{1}{2}}$$

$$\mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \\ 1 \end{pmatrix},\quad \mathbf{Q} = \begin{pmatrix} q_1 \\ q_2 \\ 1 \end{pmatrix} \in V_0$$

Isometric sphere

For $B \in PU(1,2;\mathbb{C})$; $B(\infty) \neq \infty$

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}$$

$$I(B) = \left\{ z \in \overline{H_{\mathbb{C}}^2} \;\middle|\; \rho(z, B^{-1}(\infty)) = \sqrt{\frac{2}{|e-f+h-j|}} \right\}$$

$$R_B = \sqrt{\frac{2}{|e-f+h-j|}} \quad : \text{ radius of } I(B)$$

- $R_B^2 = r_B \qquad (n=1)$

- $\rho(B(z), B(w))$

$$= \frac{R_B^2}{\rho(z, B^{-1}(\infty))\, \rho(w, B^{-1}(\infty))}\, \rho(z, w)$$

for $z, w \in \partial H_{\mathbb{C}}^2$, $B(\infty) \neq \infty$

Shimizu's Lemma

for complex hyperbolic space

G : discrete subgroup of $PU(1,2;\mathbb{C})$

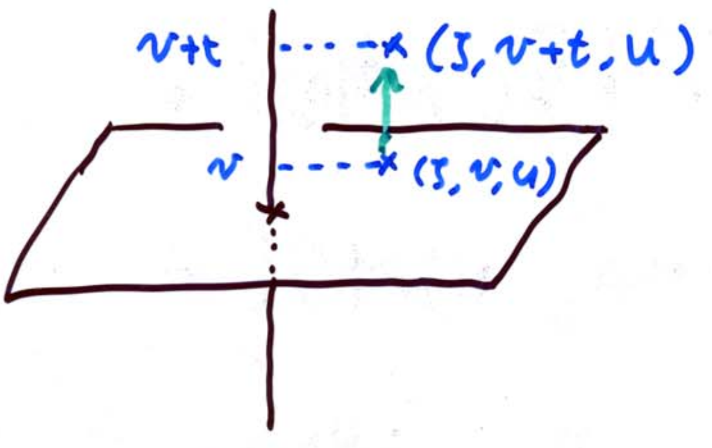
$G_\infty = \langle A \rangle$; A (parabolic)

$B \in G$; $B(\infty) \neq \infty$

① A : vertical translation

$$A_{(0,t)} : (\zeta, v, u)_H \longrightarrow (\zeta, v+t, u)_H$$

$$A_{(0,t)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+\frac{t}{2}i & \frac{t}{2}i \\ 0 & -\frac{t}{2}i & 1-\frac{t}{2}i \end{bmatrix} \quad (t \neq 0)$$



$$\Longrightarrow \; R_B^2 \leqq \rho(AB(\infty), B(\infty))\,\rho(AB^{-1}(\infty), B^{-1}(\infty)) = t$$

$U = \{(\zeta, v, u)_H \mid u > t\}$ is precisely invariant under G_∞

② A : Heisenberg translation

$A_{(\tau,t)} : (\zeta, v, u)_H$

$\longrightarrow (\tau + \zeta,\ t + v + 2\,\mathrm{Im}(\tau\bar{\zeta}),\ u)_H$

$$A_{(\tau,t)} = \begin{bmatrix} 1 & \tau & \tau \\ -\bar{\tau} & 1 - \frac{|\tau|^2 - it}{2} & -\frac{|\tau|^2 - it}{2} \\ \bar{\tau} & \frac{|\tau|^2 - it}{2} & 1 + \frac{|\tau|^2 - it}{2} \end{bmatrix}$$

$\Longrightarrow R_B^2 \leqq \rho(AB(\infty), B(\infty))\,\rho(AB^{-1}(\infty), B^{-1}(\infty))$

$+ 4|\tau|^2$

$U = \{ z = (\zeta, v, u)_H \mid$

$u > \rho(z, Az)^2 + 8|\tau|^2 \}$

precisely invariant under G_∞

③ A: screw parabolic (not unipotent)

$$A : (\zeta, v, u)_H \longrightarrow (e^{i\theta}\zeta, v+t, u)_H$$

$$A = \begin{bmatrix} e^{i\theta} & 0 & 0 \\ 0 & 1+\frac{t}{2}i & \frac{t}{2}i \\ 0 & -\frac{t}{2}i & 1-\frac{t}{2}i \end{bmatrix}$$

$$|e^{i\theta}-1| < \frac{1}{4}, \quad t\sin\theta > 0$$

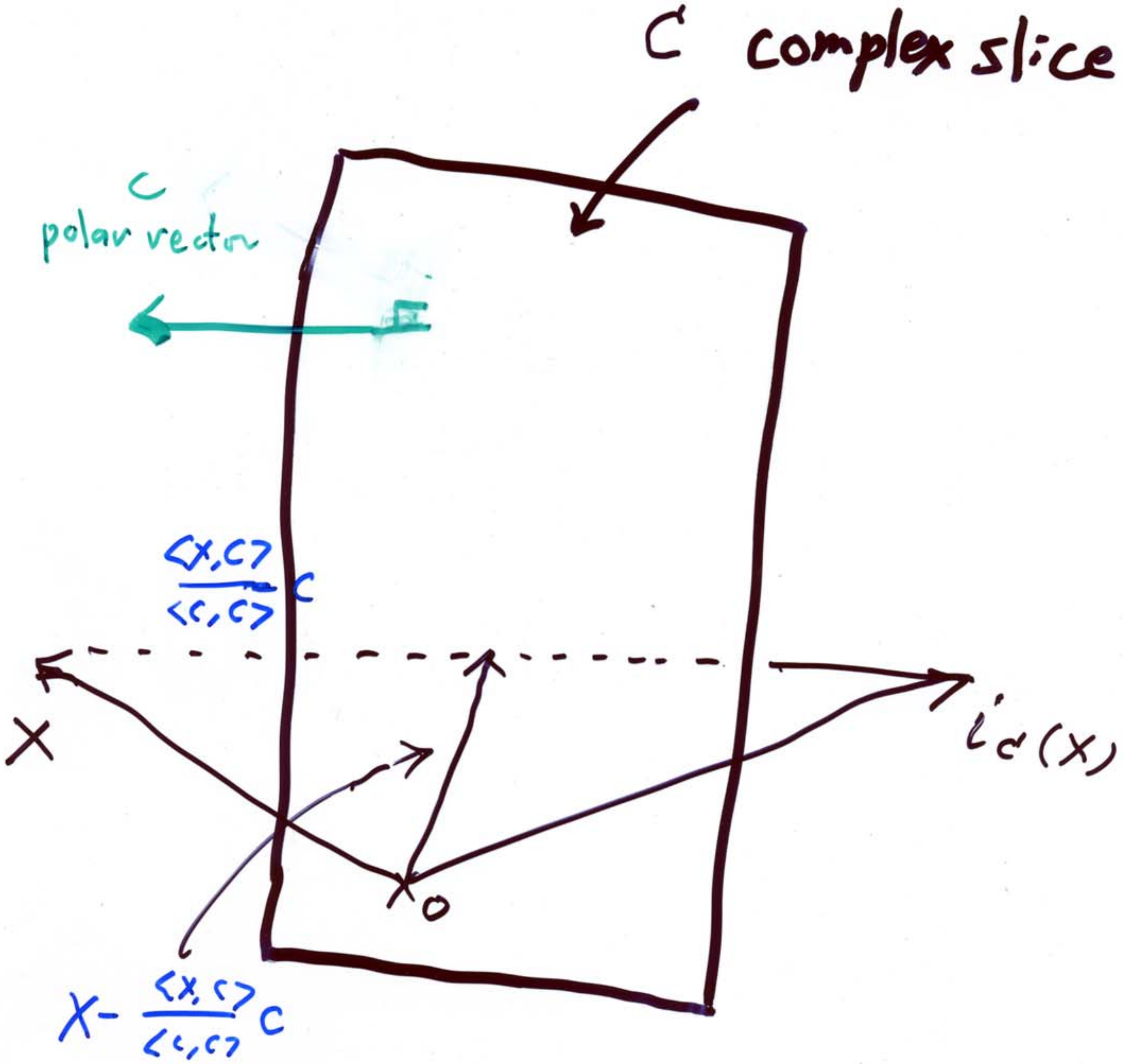
$$\Longrightarrow \quad R_B^2 \leq \frac{\rho(AB(\infty), B(\infty))\,\rho(AB^{-1}(\infty), B^{-1}(\infty))}{\delta^2}$$

$$\delta = \frac{1+\sqrt{1-4|e^{i\theta}-1|}}{2}$$

$$U = \left\{ (\zeta, v, u)_H \;\middle|\; u > \frac{2\,|2|\zeta|^2(e^{i\theta}-1)+it|}{1-6|e^{i\theta}-1|+\sqrt{1-4|e^{i\theta}-1|}} \right\}$$

$$\left(|e^{i\theta}-1| < \frac{2}{9}, \; t\sin\theta > 0\right)$$

precisely invariant under G_∞



$$i_c(x) = x - \frac{\langle x, c\rangle}{\langle c, c\rangle} c - \frac{\langle x, c\rangle}{\langle c, c\rangle} c$$

$$= x - \frac{2\langle x, c\rangle}{\langle c, c\rangle} c$$

complex reflection (fixing C)

Complex hyperbolic triangle groups

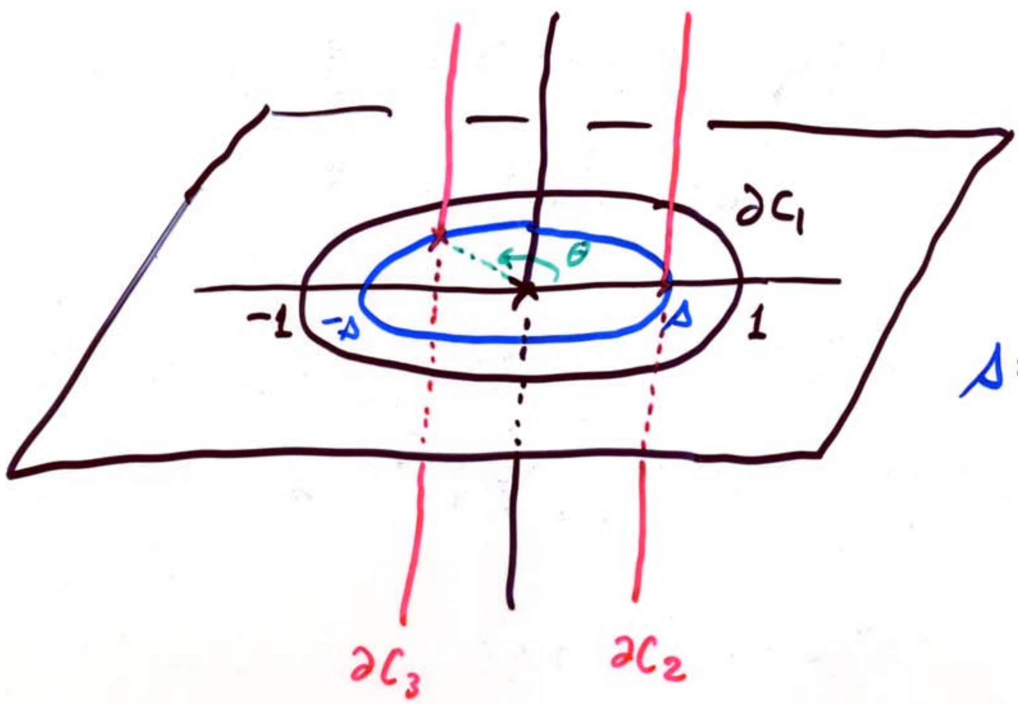
C_1, C_2, C_3: complex slices in $H^2_{\mathbb{C}}$

$$\begin{cases} \measuredangle(C_1, C_2) = \measuredangle(C_1, C_3) = \frac{\pi}{n} & (n \geq 3) \\ \overline{C_2} \cap \overline{C_3} = \{pt\} \in \partial H^2_{\mathbb{C}} \end{cases}$$

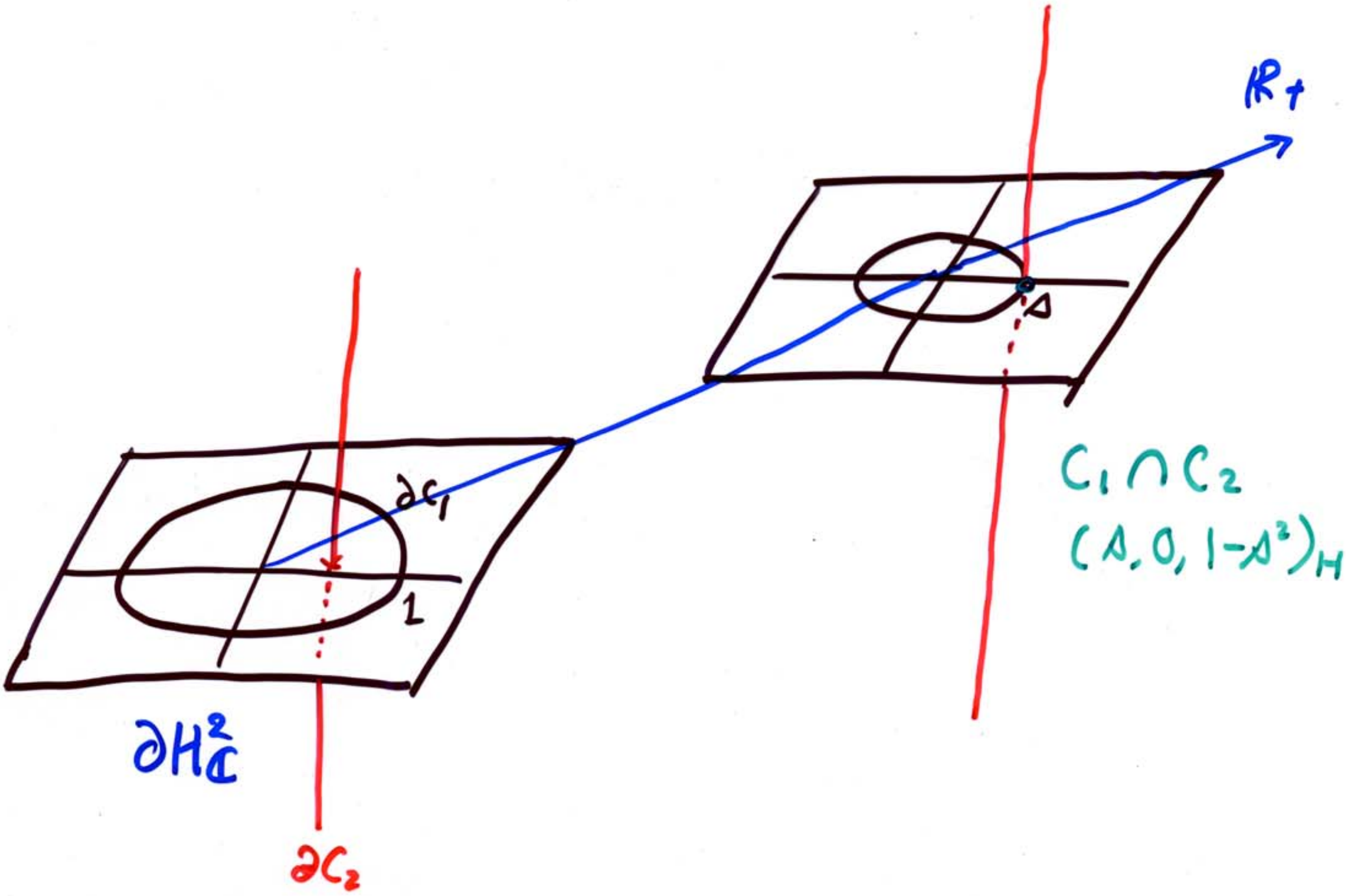
ι_j : complex reflection
(fixing a complex slice C_j)

$\Gamma = \langle \iota_1, \iota_2, \iota_3 \rangle$

complex hyperbolic triangle group of type (n, n, ∞)



$s = \cos\frac{\pi}{n}$



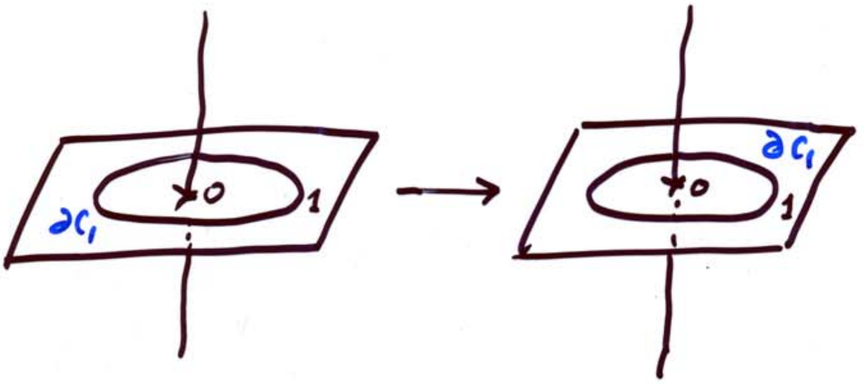
$\mathbb{R}_+$
∂C_1
1
s
$C_1 \cap C_2$
$(s, 0, 1-s^2)_H$
$\partial H^2_{\mathbb{C}}$
∂C_2

$$i_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$(\zeta, v, u)_H$$

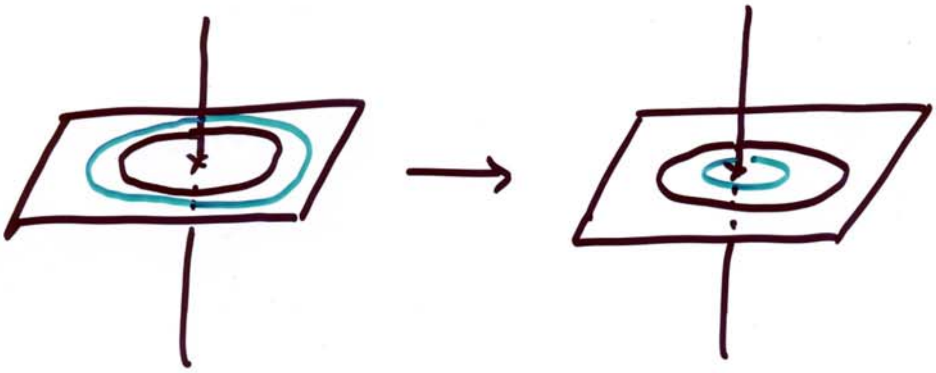
$$\longrightarrow \left(\frac{\zeta\{(|\zeta|^2+u)+iv\}}{(|\zeta|^2+u)^2+v^2}, \frac{-v}{(|\zeta|^2+u)^2+v^2}, \frac{u}{(|\zeta|^2+u)^2+v^2} \right)_H$$

1)



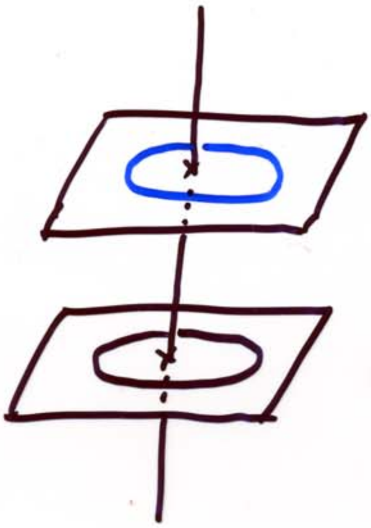
$(v=0)$

2)



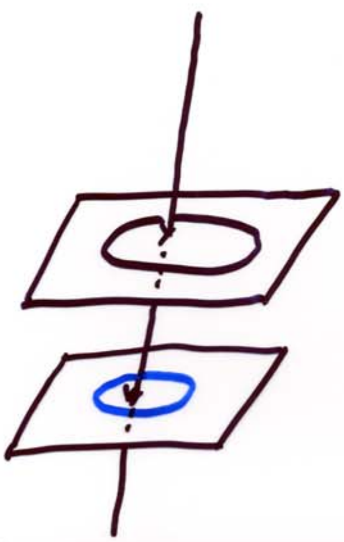
$(v=0)$

3)



$(v=1)$

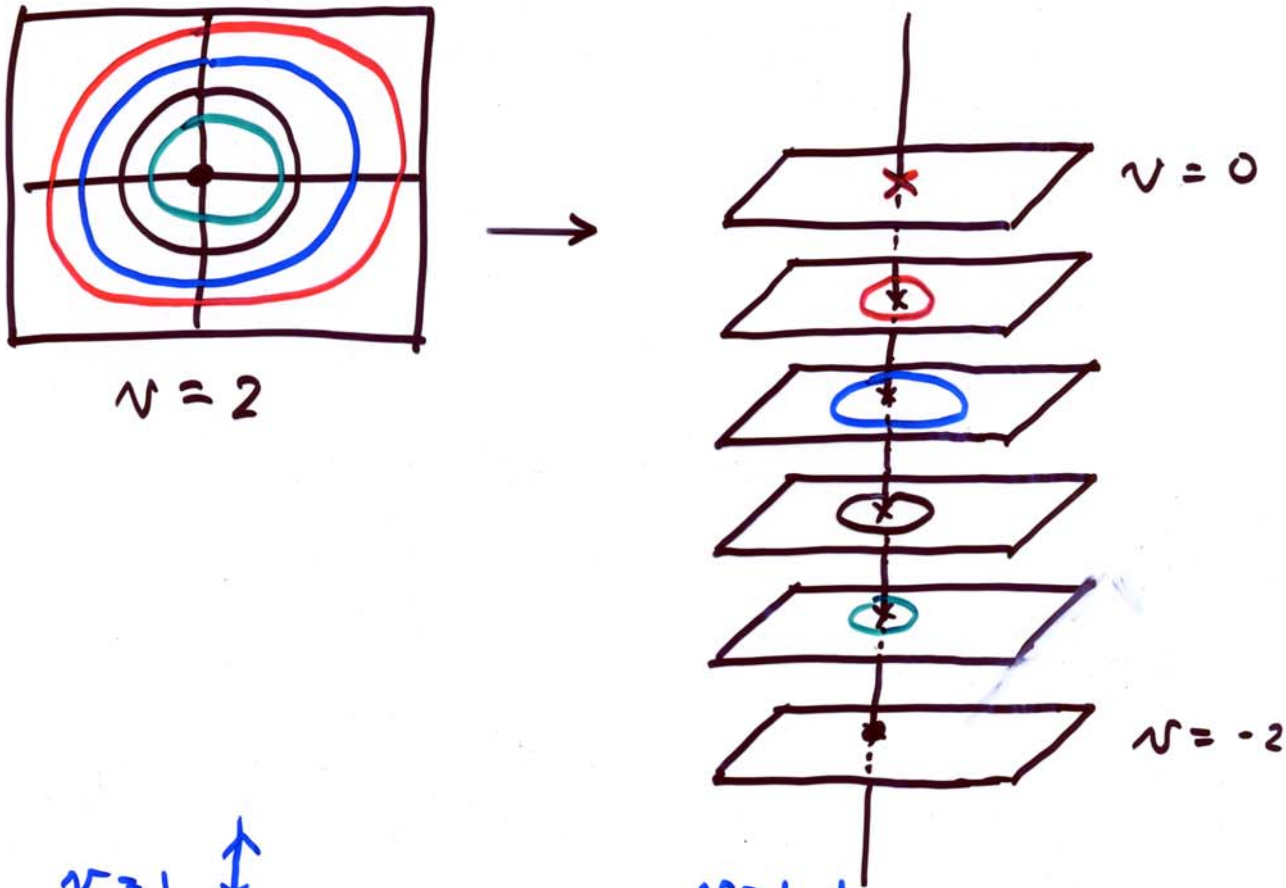
$(v=0)$



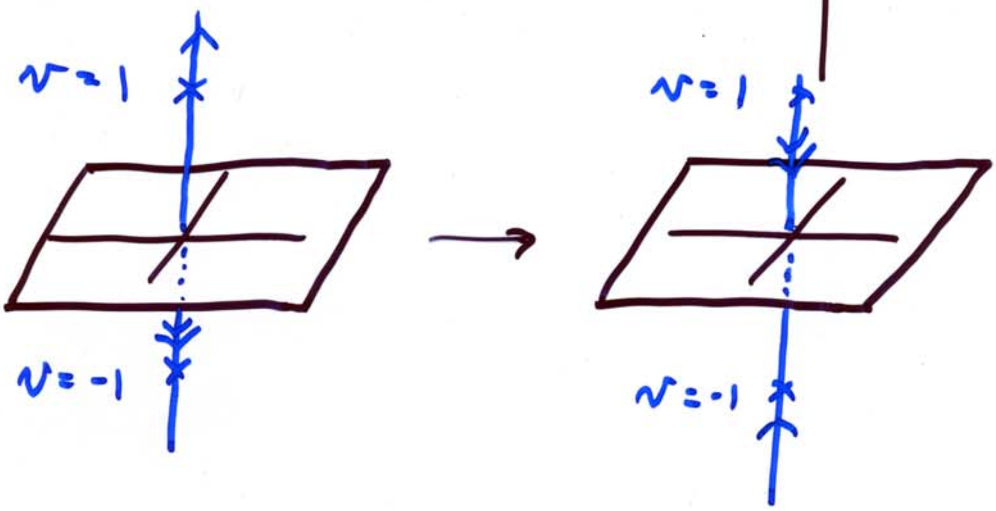
$(v=0)$

$(v=-\frac{1}{2})$

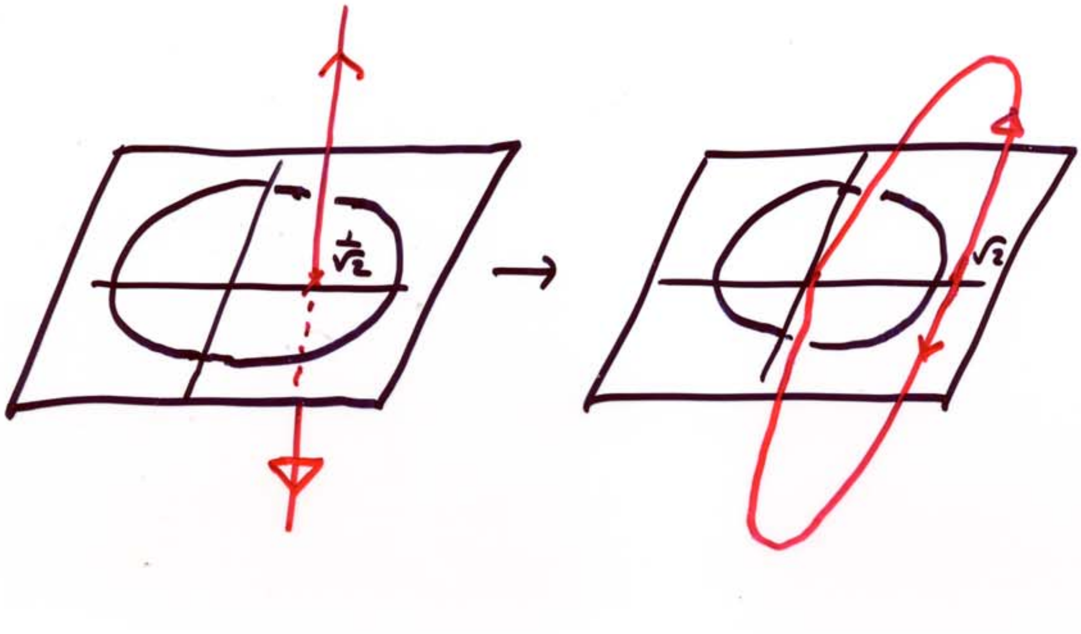
4)



5)



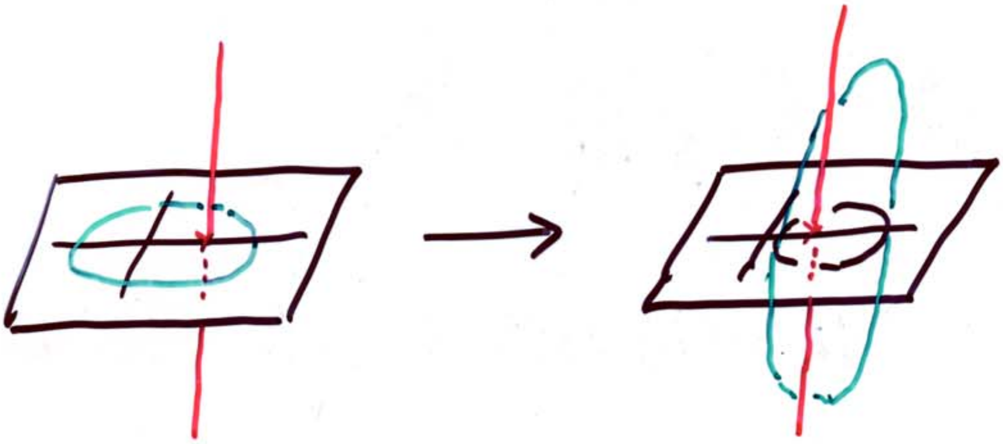
6)

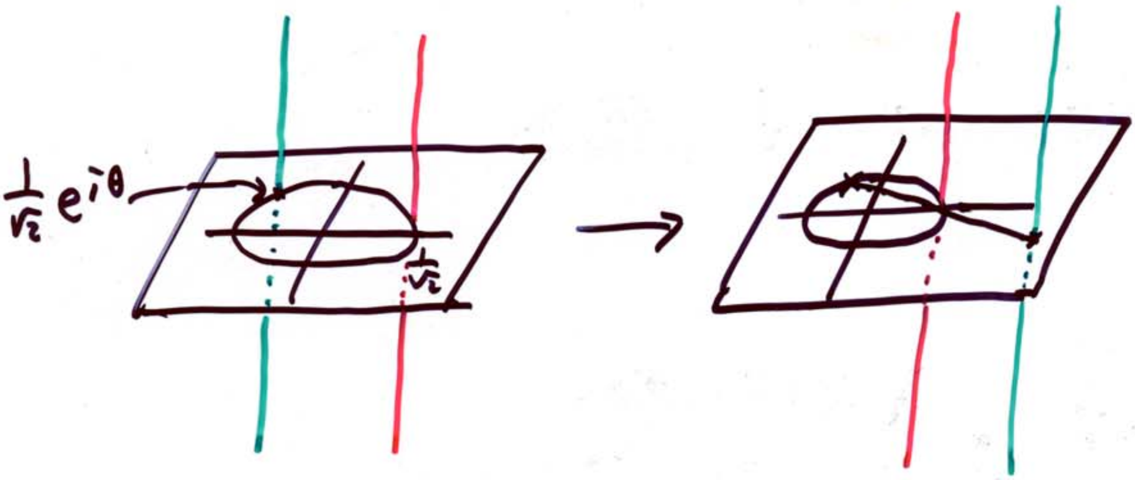


$$i_2 = \begin{bmatrix} 1 & -2s & -2s \\ -2s & 2s^2-1 & 2s^2 \\ 2s & -2s^2 & -2s^2-1 \end{bmatrix} \qquad \left(s = \cos\frac{\pi}{n}\right)$$

$n = 4$

$$(\zeta, v, u)_H \longmapsto (-\zeta + \sqrt{2},\ 2\sqrt{2}\,\mathrm{Im}(\zeta) + v,\ u)_H$$





$$i_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad i_2 = \begin{bmatrix} 1 & -2s & -2s \\ -2s & 2s^2-1 & 2s^2 \\ 2s & -2s^2 & -2s^2-1 \end{bmatrix}$$

$$i_3 = \begin{bmatrix} 1 & -2s e^{i\theta} & -2s e^{i\theta} \\ -2s e^{-i\theta} & 2s^2-1 & 2s^2 \\ 2s e^{-i\theta} & -2s^2 & -2s^2-1 \end{bmatrix}$$

$$i_2 i_3 = \begin{bmatrix} 1 & +2s(1-e^{i\theta}) & +2s(1-e^{i\theta}) \\ -2s(1-e^{-i\theta}) & 4s^2(e^{i\theta}-1)+1 & +4s^2(e^{i\theta}-1) \\ +2s(1-e^{-i\theta}) & -4s^2(e^{i\theta}-1) & -4s^2(e^{i\theta}-1)+1 \end{bmatrix}$$

$$i_1 i_2 i_3 = \begin{bmatrix} -1 & -2s(e^{i\theta}-1) & -2s(e^{i\theta}-1) \\ -2s(e^{-i\theta}-1) & 4s^2(e^{i\theta}-1)+1 & 4s^2(e^{i\theta}-1) \\ -2s(e^{-i\theta}-1) & 4s^2(e^{i\theta}-1) & 4s^2(e^{i\theta}-1)-1 \end{bmatrix}$$

$$\tau = \operatorname{trace}(i_1 i_2 i_3) = 8s^2(e^{i\theta}-1) - 1$$

$$\begin{aligned} f(\tau) = {} & 256 s^4(1-\cos\theta)^2(8s^2+1) \\ & + 320 s^2(1-\cos\theta)(8s^2+1) \\ & - 8(8s^2\cos\theta - 8s^2 - 1)^3 \\ & + 1536 s^4(8s^2\cos\theta - 8s^2 - 1)(1-\cos^2\theta) \\ & - 8 \end{aligned}$$

- $i_1 i_2 i_3$: regular elliptic

$\Longrightarrow$ $\Gamma = \langle i_1, i_2, i_3 \rangle$ is not discrete

(Pratoussevitch)

n	α_n	β_n	γ_n
3	——	——	0.9523
4	——	——	0.9833
5	——	——	0.9892
6	——	——	0.9914
7	——	——	0.9924
8	——	——	0.9930
9	0.9312	0.9319	0.9934
10	0.9367	0.9423	0.9937
11	0.9403	0.9510	0.9939
12	0.9427	0.9580	0.9940
13	0.9445	0.9637	0.9941
14	0.9458	0.9684	0.9942
15	0.9469	0.9722	0.9943
16	0.9477	0.9754	0.9943
17	0.9484	0.9781	0.9944
18	0.9489	0.9804	0.9944
19	0.9494	0.9823	0.9944
20	0.9498	0.9840	0.9945
50	0.9527	0.9973	0.9947

- $\alpha_n < \cos\theta < \beta_n \Longrightarrow i_1 i_2 i_3$: regular elliptic

  $\Longrightarrow \Gamma = \langle i_1, i_2, i_3 \rangle$: <u>not</u> discrete

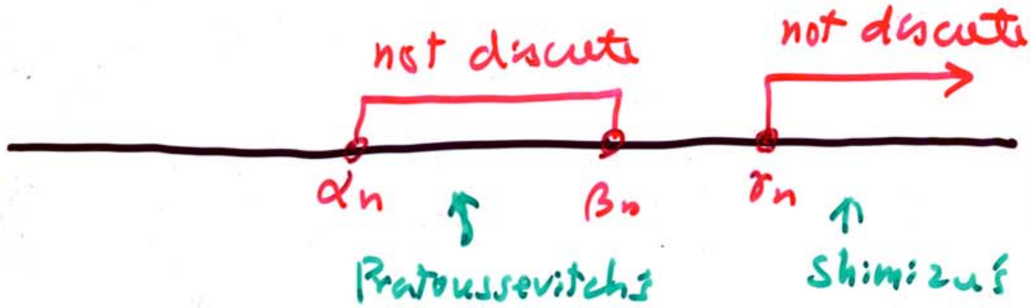
  Pratoussevitch's

- $\gamma_n < \cos\theta < 1 \Longrightarrow \Gamma = \langle i_1, i_2, i_3 \rangle$: <u>not</u> discrete

  Shimizu's

① $3 \leqq n \leqq 8$

② $9 \leqq n \leqq 34$



③ $n \geqq 35$