35th **Symposium on Transformation Groups**

November 11–13, 2008
Okayama, Japan

November 12, 2008
16:10 – 17:00

Current Trends in the Study of Smith Equivalent Representations

Krzysztof Pawałowski
Adam Mickiewicz University
Poznań, Poland

Toshio Sumi
Kyushu University
Fukuoka, Japan

In 1960, Paul A. Smith has asked the question: *If a finite group G acts smoothly on a sphere with exactly two fixed points, is it true that the tangent G-modules at the two fixed points are isomorphic to each other?*

Let G be a finite group. Two real G-modules U and V are called *Smith equivalent* if

$$U \cong T_a(S) \text{ and } V \cong T_b(S)$$

for a smooth action of G on a homotopy sphere S with exactly two fixed points a and b.

In the real representation ring $RO(G)$ of G, we consider the subset $Sm(G)$ consisting of the differences $U - V$ of real G-modules U and V which are Smith equivalent.

As $U - V = 0$ in $RO(G)$ if and only if $U \cong V$ as real G-modules, the Smith question can be restated as follows:

Is it true that $Sm(G) = 0$?

Atiyah and Bott: $Sm(\mathbb{Z}_p) = 0$ *for any prime* p.

Sanchez: $Sm(\mathbb{Z}_{p^k}) = 0$ *for any odd prime* p *and any integer* $k \geq 1$.

Character theory: $Sm(S_3) = 0$, $Sm(\mathbb{Z}_n) = 0$ *for* $n = 2$, 4, *or* 6.

Cappell and Shaneson, Petrie: $Sm(\mathbb{Z}_n) \neq 0$ *for* $n = 4q$ *with* $q \geq 2$. *In particular,* $Sm(\mathbb{Z}_8) \neq 0$.

Petrie: $Sm(G) \neq 0$ *for a finite abelian group* G *of odd order, containing four or more noncyclic Sylow subgroups*.

Dovermann and Petrie: $Sm(G) \neq 0$ *for large families of cyclic groups* G *of odd order*.

Masuda, Cho, Suh, Illman, and others obtained many results related to the Smith question.

During the past 10 years, more answers were given by Laitinen, Pawałowski, Solomon, Sumi, Morimoto, Ju, Koto, and Qi.

Two real G-modules U and V, which are Smith equivalent, are isomorphic when restricted to any cyclic subgroup of G of order 2, 4, and of odd prime power order.

In 1996, Laitinen suggested to study the Smith isomorphic question under the condition that for the corresponding action of G on S, the fixed point set S^g is connected for any element $g \in G$ of order 2^k for $k \geq 3$, and thus U and V are isomorphic when restricted to any (cyclic) subgroup of G of prime power order.

For a finite group G not of prime power order, let $LSm(G) \subseteq Sm(G)$ consist of the differences of Smith equivalent real G-modules U and V such that the Laitinen condition holds.

Then $LSm(G) \subseteq PO(G)$, the free abelian group of the differences in $RO(G)$ of real G-modules U and V such that $\text{Res}^G_P(U) \cong \text{Res}^G_P(V)$ for any subgroup P of G of prime power order, and $\dim U^G = \dim V^G = 0$.

Let r_G be the number of real conjugacy classes of elements in G not of prime power order.

For $r_G = 0$ or 1, $PO(G) = 0$ and for $r_G \geq 2$, rank $PO(G) = r_G - 1$. In particular,

$$PO(G) \neq 0 \text{ if and only if } r_G \geq 2.$$

Laitinen Conjecture. Let G be a finite Oliver group. Then $LSm(G) \neq 0$ if and only if $r_G \geq 2$.

For any finite perfect group G, the conjecture is true by Laitinen and Pawałowski (1999), and moreover, $LSm(G) = PO(G)$ by Pawałowski and Solomon (2002).

As $LSm(G) \subseteq PO(G)$, the condition $r_G \geq 2$ is necessary in the Laitinen Conjecture.

Morimoto proves for the first time that this condition is not sufficient for $G = \mathrm{Aut}(A_6)$, by computing that $Sm(G) = 0$ while $r_G = 2$.

Pawałowski and Sumi obtain similar results for a number of finite solvable Oliver groups G, including the affine group $G = \mathrm{Aff}(2,3)$.

Let G^{nil} (resp. G^{sol}) be the smallest normal subgroup of G such that G/G^{nil} is nilpotent (resp. G/G^{sol} is solvable). Recall that

$$G^{\mathsf{nil}} = \bigcap_p O^p(G)$$

where $O^p(G)$ is the smallest normal subgroup of G such that $G/O^p(G)$ is a p-group.

Definition. We say that a finite group G is of *nil-type* (resp. *sol-type*), if there exist two elements x and y of G such that the following three conditions hold.

(1) $xG^{\mathsf{nil}} = yG^{\mathsf{nil}}$ (resp. $xG^{\mathsf{sol}} = yG^{\mathsf{sol}}$).

(2) x and y are not real conjugate in G, and the orders $|x|$ and $|y|$ are not prime powers (and thus $r_G \geq 2$).

(3) x and y are in some gap subgroup of G, or $|x|$ and $|y|$ are even and the involutions of $\langle x \rangle$ and $\langle y \rangle$ are conjugate in G.

A finite group G is called an *Oliver group* if the following algebraic condition holds: *there does not exist* a series of normal subgroups $P \trianglelefteq H \trianglelefteq G$ such that P is a p-group and G/H is a q-group for some primes p and q, possibly $p = q$, and H/P is cyclic.

According to Oliver, this algebraic condition is necessary and sufficient for G to have a smooth fixed point free action on a disk, or as proven by Laitinen and Morimoto, to have a smooth one fixed point action on a sphere.

For a finite group G, let $\mathcal{P}(G)$ be the family of subgroups of G of prime power order, and $\mathcal{L}(G)$ be the family of large subgroups of G, where a subgroup L of G is called *large in* G if $O^p(G) \leq L$ for some prime p.

A finite group G is called a *gap group* if

$$\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$$

and there exists a real G-module W such that $\dim W^P > 2 \dim W^H$ for all $P < H \leq G$ with $P \in \mathcal{P}(G)$, and $\dim W^L = 0$ for all $L \in \mathcal{L}(G)$.

Examples of classes of finite Oliver groups

$$\{\text{simple} \ncong \mathbb{Z}_p\} \subset \{\text{perfect}\} \subset$$

$$\{\text{nonsolvable gap}\} \subset$$

$$\{\text{nonsolvable}\} \subset \{\text{Oliver}\}$$

$$\{\text{abelian}_{(3)}\} \subset \{\text{nilpotent}_{(3)}\} \subset$$

$$\{\text{solvable Oliver}/N \cong \mathbb{Z}_{pq}\} \ \cup$$

$$\{\text{solvable Oliver odd order}\}$$

$$\subset \{\text{solvable Oliver gap}\}$$

The index (3) above means that G has three or more noncyclic Sylow subgroups, Moreover, p and q are two distinct odd primes.

As Pawałowski and Solomon prove in 2002, for a finite Oliver group G, $LSm(G) \neq 0$ when $|G|$ is odd, or when G has a quotient isomorphic to $\mathbb{Z}_{pq}$ for two distinct odd primes p and q.

Theorem A. *Let G be a finite Oliver group of nil type. Then $LSm(G) \neq 0$.*

Theorem B. *Except for $G = \text{Aut}(A_6)$ and $P\Sigma L(2,27)$, a finite nonsolvable group G is of sol-type (equiv. nil-type) if and only if $r_G \geq 2$.*

Theorem C. *Except for $G = \text{Aut}(A_6)$, the following claim holds for any finite nonsolvable group G: $LSm(G) \neq 0$ if and only if $r_G \geq 2$.*

The claim of Theorem C has been proven by Laitinen and Pawałowski for any finite perfect group G, and by Pawałowski and Solomon for any finite nonsolvable gap group G, except for $G = P\Sigma L(2,27)$.

By our definition, $P\Sigma L(2,27)$ is a semi-direct product of $PSL(2,27)$ and $\mathrm{Aut}(\mathbb{F}_{27})$.

The two nonsolvable groups $G = \mathrm{Aut}(A_6)$ and $G = P\Sigma L(2,27)$ are not of nil-type. In both cases, $r_G = 2$. Moreover, $\mathrm{Aut}(A_6)$ is not a gap group while $P\Sigma L(2,27)$ is a gap group.

As Morimoto shows, $Sm(\mathrm{Aut}(A_6)) = 0$ and $LSm(P\Sigma L(2,27)) \cong \mathbb{Z}$.

Hence, Theorems A and B yield Theorem C. Remember, $LSm(G) = 0$ for $r_G = 0$ or 1.

Later, we will give some ideas about the proofs of Theorems A and B.

Now, we recall that by the work of Atiyah and Bott from 1968, $Sm(\mathbb{Z}_p) = 0$ for any prime p, and by the work of Pawałowski and Solomon from 2002, the following theorem holds.

Theorem D. *Let G be a finite simple group, and assume that G is nonabelian, i.e. $G \ncong \mathbb{Z}_p$. Then $Sm(G) = 0$ if and only if $r_G = 0$ or 1.*

There are exactly fourteen finite nonabelian simple groups G with $r_G = 0$ or 1:

$r_G = 0$: $G = PSL(2, q)$ *with* $q = 5, 7, 8, 9, 17$, $PSL(3, 4)$, $Sz(8)$, $Sz(32)$, *and*

$r_G = 1$: $PSL(2, 11)$, $PSL(2, 13)$, $PSL(3, 3)$, A_7, M_{11}, M_{22}.

In these fourteen cases, it has been checked that $LSm(G) = Sm(G)$, and thus $Sm(G) = 0$.

Only four of the groups G above have elements of order 8: $PSL(2, 17)$, $PSL(3, 3)$, M_{11}, M_{22}, and in each of the case, $\dim W^g > 0$ for any irreducible real G-module W and any element $g \in G$ of order 2^k for $k \geq 3$.

Theorem E. *In either of the following cases, $Sm(G) = 0$ if and only if $r_G = 0$ or 1.*

(1) $G = PSL(n, q)$ *or* $SL(n, q)$ *for* $n \geq 2$ *and every prime power* q.

(2) $G = PSp(n, q)$ *or* $Sp(n, q)$ *for* $n \geq 2$, n *even, and every prime power* q.

(3) $G = A_n$ *or* S_n *for* $n \geq 2$.

(4) $G = PGL(n, q)$ *or* $GL(n, q)$ *for* $n \geq 2$ *and every prime power* q.

(5) $G = \mathrm{Aff}(n, q)$ *for* $n \geq 2$, *and every prime power* q *with* $(n, q) \neq (2, 3)$.

In Theorem E, $r_G = 0$ or 1 only for small values of n and q, and in every case,

$$LSm(G) = Sm(G) = 0$$

by representation theory arguments.

According to Pawałowski and Sumi (2008), for $G = \mathrm{Aff}(2,3)$, $Sm(G) = 0$ while $r_G = 2$.

In Theorem E, the cases (1)–(3) were already covered by Pawałowski and Solomon, using the equivariant surgery under the gap condition to prove that $LSm(G) \neq 0$ for $r_G \geq 2$.

To prove the same conclusion in the cases (4) and (5), we need to apply the equivariant surgery under the weak gap condition.

Let G be a finite group. For a real G-module or a smooth G-manifold X, set

$$d_X(P,H) = \dim X^P - 2\dim X^H$$

for any series $P < H \le G$ of subgroups of G, where $P \in \mathcal{P}(G)$. We say that X satisfies the *weak (resp. semi-weak) gap condition* if the conditions (1)–(4) (resp. (1) and (2)) hold.

(1) $d_X(P,H) \ge 0$ for every pair (P,H).

(2) If $d_X(P,H) = 0$ for some pair (P,H), then $[H : P] = 2$ and X^H is connected with $\dim X^H > \dim X^K + 1$ for every $H < K \le G$.

(3) If $d_X(P,H) = 0$ for some pair (P,H), and $[H : P] = 2$, then X^H can be oriented in such a way that the map $g\colon X^H \to X^H$ is orientation preserving for any $g \in N_G(H)$.

(4) If $d_X(P,H) = d_X(P,H') = 0$ for two pairs (P,H) and (P,H'), then $\langle H,H'\rangle \notin \mathcal{L}(G)$.

Let $PO(G)^{\mathcal{L}}$ consist of the differences $U - V$ in $PO(G)$ such that $\dim U^L = \dim V^L = 0$ for $L \in \mathcal{L}(G)$, the family of large subgroups of G.

Let $PO(G)^{\mathcal{L}}_{\mathsf{wg}}$ (resp. $PO(G)^{\mathcal{L}}_{\mathsf{sg}}$) be defined by the additional restriction that U and V satisfy the weak (resp. semi-weak) gap condition.

S-Theorem. *Let G be a finite Oliver group. Then $2PO(G)^{\mathcal{L}}_{\mathsf{sg}} \subseteq PO(G)^{\mathcal{L}}_{\mathsf{wg}} \subseteq LSm(G)$.*

R-Theorem. *Let G be a finite Oliver group of nil-type. Then $PO(G)^{\mathcal{L}}_{\mathsf{sg}} \neq 0$.*

Theorem A asserts that $LSm(G) \neq 0$ for any finite Oliver group G of nil-type. This claim is true by the two theorems posed above.

Remark. If G is a finite gap group, then

$$PO(G)^{\mathcal{L}}_{\mathsf{sg}} = PO(G)^{\mathcal{L}}_{\mathsf{wg}} = PO(G)^{\mathcal{L}}$$

and thus $PO(G)^{\mathcal{L}} \subseteq LSm(G)$ for any finite Oliver gap group G. If G is a perfect group, then $PO(G)^{\mathcal{L}} = PO(G) = LSm(G)$.

For $H \trianglelefteq G$, $PO(G,H)$ consist of the differences $U - V$ in $PO(G)$ of real G-modules U and V such that $U^H \cong V^H$ as real G/H-modules.

The rank of the free abelian group $PO(G,H)$ has been computed to the effect that

$$\operatorname{rank} PO(G,H) = r_G - \overline{r}_{G/H}$$

where $\overline{r}_{G/H}$ is the number of real conjugacy classes in G/H of cosets containing elements $g \in G$ not of prime power order. Therefore

$$PO(G,H) \neq 0 \text{ if and only if } r_G > \overline{r}_{G/H}.$$

For a finite Oliver group G, the following two conclusions are true.

(1) If G is of nil-type, then $r_G > \overline{r}_{G/G^{\mathrm{nil}}}$.

(2) If G is a gap group and $r_G > \overline{r}_{G/G^{\mathrm{nil}}}$, then G is of nil-type.

Corollary. *A finite Oliver gap group G is of nil-type if and only if $r_G > \overline{r}_{G/G^{\mathrm{nil}}}$.*

Proposition. *Except for* $G = \mathrm{Aut}(A_6)$ *and* $P\Sigma L(2,27)$*, the following three conditions are equivalent for any finite nonsolvable group* G*.*

(1) $r_G > \overline{r}_{G/G^{\mathrm{sol}}}$*, i.e.* $PO(G, G^{\mathrm{sol}}) \neq 0$*.*

(2) $r_G \geq 2$*.*

(3) G *is of sol-type.*

The equivalence of (1) and (2) is obtained by Pawałowski and Solomon. By using a similar approach and some technical computations, we show that (1) and (3) are also equivalent, proving Theorem B asserting that (2) and (3) are equivalent.