A New Theorem to Find Smith Equivalent Representations

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§ Our Problem

There are (finite) Oliver groups G such that

$$\mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} = 0$$

but

$$\mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)} \neq 0.$$

Examples:

(1) $P\Sigma L(2,27)$ (Pawałowski-Solomon)

(2) $SG(864,2666)$ (Sumi)

(3) $SG(864,4666)$ (Sumi)

We ask whether $Sm(G) \neq 0$ for such a group G.

$\mathrm{RO}(G)$: real representation ring

For $A \subset \mathrm{RO}(G)$, $\mathcal{F}$, $\mathcal{G} \subset \mathcal{S}(G)$,

$$
\begin{aligned}
A_{\mathcal{F}} &:= \{V - W \in A \mid \mathrm{res}_H V \cong \mathrm{res}_H W \quad (\forall H \in \mathcal{F})\}, \\
A^{\mathcal{G}} &:= \{V - W \in A \mid V^K = 0 = W^K \quad (\forall K \in \mathcal{G})\}, \\
A^{\mathcal{G}}_{\mathcal{F}} &:= (A^{\mathcal{G}})_{\mathcal{F}}.
\end{aligned}
$$

Thus,

$$\mathrm{RO}(G)_{\mathcal{P}(G)} = \{V - W \mid V,\ W\text{: } \mathcal{P}(G)\text{-matched}\},$$

$$\mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)} = \{V - W \mid V,\ W\text{: } \mathcal{P}(G)\text{-matched, } \underline{\text{type } 0}\}$$

$$\underline{\dim V^G = 0 = \dim W^G}.$$

p: a prime

$G^{\{p\}}$ (Dress subgroup): the smallest normal subgroup of G s.t.

$$G/G^{\{p\}} \text{ has } p\text{-power order.}$$

$\mathcal{L}(G)$ is the set of 'large subgroups' of G, i.e.

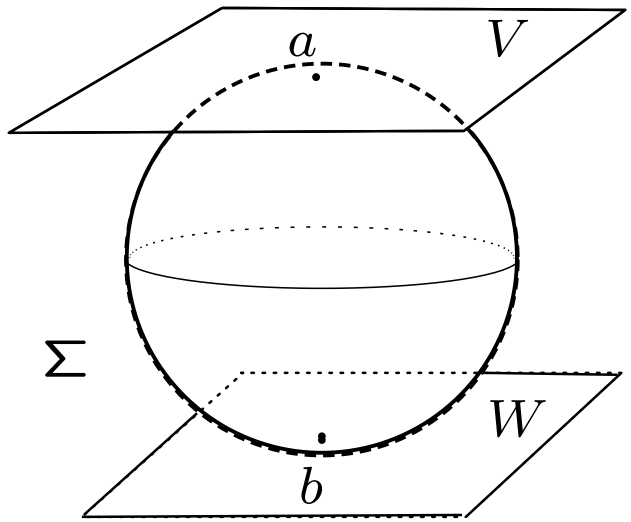
$$\mathcal{L}(G) := \{L \leq G \mid L \geq G^{\{p\}} \text{ for some } p\}.$$

G^{nil}: the smallest normal subgroup of G s.t.

$$G/G^{\mathsf{nil}} \text{ is nilpotent.}$$

$V \sim_{\mathfrak{S}} W$ (**Smith equivalent**) $\Leftrightarrow \exists$ homotopy sphere $\boldsymbol{\Sigma}$ s.t.

$$\Sigma^G = \{a, b\}, \quad T_a(\Sigma) = V, \quad T_b(\Sigma) = W.$$



$$Sm(G) := \{V - W \in \mathrm{RO}(G) \mid V \sim_{\mathfrak{S}} W\}.$$

Smith equivalent pairs (V, W) obtained by various authors satisfy

$$V^N = 0 = W^N$$

for all $N \triangleleft G$ s.t.

$$|G/N| = \text{prime}.$$

Prob. We would like to find Smith equivalent pairs (V, W) s.t.

$$V^N \neq 0 \quad \text{and} \quad W^N = 0$$

for some $N \triangleleft G$ such that $|G/N| = p$ (some odd prime p).

Sumi's Prob. T. Sumi (06) checked whether $Sm(G)_{\mathcal{P}(G)} = 0$ or not for G (order ≤ 2000) $\notin$ List 1.

$SG(m,n)$	a_G	gap?	G/G^{nil}	N.B.
$SG(864, 2666)$	2	Yes	C_3	$\cdots \mathbb{Z}$
$SG(864, 4663)$	3	No	C_8	
$SG(864, 4666)$	2	Yes	C_3	$\cdots \mathbb{Z}$
$SG(864, 4672)$	5	Yes	$Q_8 \times C_3$	
$SG(1176, 220)$	2	Yes	C_3	$\cdots 0$
$SG(1176, 221)$	2	Yes	C_3	$\cdots 0$
$SG(1152, 155470)$	2	Yes	C_6	
$SG(1152, 157859)$	2	Yes	C_6	

List 1

§ Real Projective Space

Hypothesis: $G^{\mathsf{nil}} = G^{\{p\}} \neq G$ (p: an odd prime).

V: a real G-module s.t. $V^G = \mathbb{R}$. $\quad N := G^{\mathsf{nil}}$.
$M := P(V^N)$: the real projective space. Then

$$M^G = \{x\}, \quad T(M) \oplus \varepsilon_M \cong \gamma \otimes V^N,$$
$$T(M)^L \oplus \varepsilon_{M^L} \cong \gamma|_{M^L} \otimes V^L \quad (L \in \mathcal{L}(G)),$$
$$\gamma \oplus \gamma^\perp = M \times V^N.$$

W: an $\mathcal{L}(G)$-free G-module $\mathcal{P}(G)$-matched with V.
$\eta_M := (\gamma \otimes V) \oplus (\gamma^\perp \otimes W)$. Then
$\mathrm{res}_P(\eta_M) \cong_P (\gamma \oplus \gamma^\perp) \otimes V \cong_P M \times (V^N \otimes V)$.
${\eta_M}^L = \gamma|_{M^L} \otimes V^L \cong T(M)^L \oplus \varepsilon_{M^L}$ for $L \in \mathcal{L}(G)$.

§ Results

G: an Oliver group satisfying Hypothesis.

$N = G^{\mathrm{nil}} = G^{\{p\}}$.

$$\mathbb{R}[G]_{\mathcal{L}} := \mathbb{R}[G] - \mathbb{R}[G/N].$$

Lem 1. V and W: as above. a: a sufficiently large integer.

Then $\exists$ a smooth G-action on a disk D s.t.
(1) $D^G = \{x\}$,
(2) $D^L{}_x = P(V^N)^L$ (conn. component) for all $L \in \mathcal{L}(G)$,
(3) $(|\pi_1(D^Q)|, q) = 1$ for any $Q \in \mathcal{P}(G)$ and $|Q| = q^a > 1$,
(4) $T_x(D) = (V - V^G) \oplus \mathbb{R}[G]_{\mathcal{L}}{}^{\oplus a}$.

$\mathbb{R}[G/N]_G := \mathbb{R}[G/N] - \mathbb{R}[G/N]^G$.

A $\mathcal{P}(G)$-matched pair (U_1, U_2) is called of <u>type $\mathbb{R}[G/N]$</u> if

$$U_1^N \cong \mathbb{R}[G/N] \text{ and } U_2^N = 0.$$

A $\mathcal{P}(G)$-matched pair (V_1, V_2) is called of <u>type ${\mathbb{R}[G/N]_G}^n$</u> if there exists a $\mathcal{P}(G)$-matched pair (U_1, U_2) of type $\mathbb{R}[G/N]$ such that

$$V_1^N = (U_1^N - U_1^G)^{\oplus n} \text{ and } V_2^N = 0.$$

In the case $n = 0$, $V_1^N = 0$.

Lem 2. Suppose N has a subquotient $\cong D_{2qr}$ $(q \neq r)$.
$(V_1^{(i)}, V_2^{(i)})$: a $\mathcal{P}(G)$-matched pair of type $\mathbb{R}[G/N]_G{}^{n_i}$ with respect to $(U_1^{(i)}, U_2^{(i)})$ of type $\mathbb{R}[G/N]$ $(1 \leq i \leq m)$.
Suppose $V_1^{(1)}$, ..., $V_1^{(m)}$ are $\mathcal{P}(G)$-matched with one another.
a: a sufficiently large integer

$\Rightarrow \exists$ a smooth G-action on a disk D such that
(1) $D^G = \{x_1, \ldots, x_m\}$,
(2) $D^N{}_{x_i} = P(U_1^{(i)N})^{\times n_i}$ (conn. comp.), $1 \leq i \leq m$,
(3) $T_{x_i}(D) \cong V_1^{(i)} \oplus \mathbb{R}[G]_{\mathcal{L}}{}^{\oplus a}$.

$$\mathbb{R}[G]_{\mathcal{L}} = \mathbb{R}[G] - \mathbb{R}[G/N]$$

Thm 1. V: a gap G-module.

$$\dim V^P > 2 \dim V^H \text{ whenever } P \in \mathcal{P}(G),\ P < H$$

Suppose N has a subquotient $\cong D_{2qr}$ $(q \neq r)$.
$(V_1^{(i)}, V_2^{(i)})$: a $\mathcal{P}(G)$-matched pair of type $\mathbb{R}[G/N]_G{}^{n_i}$ with respect to $(U_1^{(i)}, U_2^{(i)})$ of type $\mathbb{R}[G/N]$ $(1 \leq i \leq m)$.
Suppose $V_1^{(1)}, \ldots, V_1^{(m)}$ are $\mathcal{P}(G)$-matched with one another.
a, b: sufficiently large integers

$\Rightarrow$ $\exists$ a smooth G-action on a sphere S such that
(1) $S^G = \{x_1, \ldots, x_m\}$,
(2) $S^N{}_{x_i} = P(U_1^{(i)N})^{\times n_i}$ (conn.comp.), $1 \leq i \leq m$,
(3) $T_{x_i}(S) \cong V_1^{(i)} \oplus \mathbb{R}[G]_{\mathcal{L}}{}^{\oplus a} \oplus V^{\oplus b}$.

Cor 1. G: Oliver group s.t.

(1) $|G/G^{\mathrm{nil}}| = p$ (p odd prime), (2) $\mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)} \neq 0$,

(3) G^{nil} has a subquotient $\cong D_{2qr}$ ($q \neq r$).

Then G is a gap group and $Sm(G) \supset \mathbb{Z}$. Particularly,

if $p = 3$ and $\mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)} \cong \mathbb{Z}$ then $Sm(G) = \mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)}$.

Cor 2. If $G = P\Sigma L(2, 27)$, $SG(864, 2666)$, or $SG(864, 4666)$ then $Sm(G) = \mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)} \cong \mathbb{Z}$.

§ Representations of $P\Sigma L(2,27)$

$G := P\Sigma L(2,27) = PSL(2,27) \rtimes \mathrm{Aut}(\mathbb{F}_{27})$.

$|PSL(2,27)| = 9828$,

$|\mathrm{Aut}(\mathbb{F}_{27})| = 3$, and

$|P\Sigma L(2,27)| = 29484$.

$PSL(2,27) = [G,G]$ the commutator subgroup.

The character table of the irreducible complex G-representaions is as follows.

	1a	6a	6b	3a	3b	2a	14a	7a	3c	3d	9a	9b	9c	9d	13a	13b
2P	1a	3a	3b	3b	3a	1a	7a	7a	3d	3c	9b	9a	9d	9c	13b	13a
3P	1a	2a	2a	1a	1a	2a	14a	7a	1a	1a	3d	3c	3c	3d	13a	13b
5P	1a	6b	6a	3b	3a	2a	14a	7a	3d	3c	9b	9a	9d	9c	13b	13a
7P	1a	6a	6b	3a	3b	2a	2a	1a	3c	3d	9a	9b	9c	9d	13b	13a
11P	1a	6b	6a	3b	3a	2a	14a	7a	3d	3c	9b	9a	9d	9c	13b	13a
13P	1a	6a	6b	3a	3b	2a	14a	7a	3c	3d	9a	9b	9c	9d	1a	1a
X.1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	1	A	/A	/A	A	1	1	1	1	1	/A	A	/A	A	1	1
X.3	1	/A	A	A	/A	1	1	1	1	1	A	/A	A	/A	1	1
X.4	13	1	1	1	1	1	1	-1	C	/C	A	/A	/A	A	.	.

X.5	13	1	1	1	1	1	1	-1	/C	C	/A	A	A	/A	.	.
X.6	13	A	/A	/A	A	1	1	-1	C	/C	1	1	A	/A	.	.
X.7	13	/A	A	A	/A	1	1	-1	/C	C	1	1	/A	A	.	.
X.8	13	A	/A	/A	A	1	1	-1	/C	C	A	/A	1	1	.	.
X.9	13	/A	A	A	/A	1	1	-1	C	/C	/A	A	1	1	.	.
X.10	27	-1	-1	3	3	-1	-1	-1	.	.	.	.	.	.	1	1
X.11	27	-/A	-A	B	/B	-1	-1	-1	.	.	.	.	.	.	1	1
X.12	27	-A	-/A	/B	B	-1	-1	-1	.	.	.	.	.	.	1	1
X.13	78	.	.	.	.	6	-1	1	-3	-3	.	.	.	.	.	.
X.14	78	.	.	.	.	-6	1	1	-3	-3	.	.	.	.	.	.
X.15	84	.	.	.	.	.	.	.	3	3	.	.	.	.	D	*D
X.16	84	.	.	.	.	.	.	.	3	3	.	.	.	.	*D	D

A = E(3)^2, B = 3*E(3)^2, C = -E(3)+2*E(3)^2,

D = E(13)^2+E(13)^5+E(13)^6+E(13)^7+E(13)^8+E(13)^11

The complete representatives of irreducible real G-representaions are $\mathbb{R}$, V_1, $\ldots$, V_{10} with characters

```
Y.0 = X.1,
Y.1 = X.2 + X.3
Y.2 = X.4 + X.5,
Y.3 = X.6 + X.7,
Y.4 = X.8 + X.9,
Y.5 = X.10,
Y.6 = X.11 + X.12,
Y.7 = X.13,
Y.8 = X.14,
Y.9 = X.15,
Y.10 = X.16, respectively.
```

Remark 1. The group $P\Sigma L(2,27)$ is isomorphic to the subgroup of S_{28}, the symmetric group on 28 letters, generated by the four elements

$$\begin{aligned}
\sigma_1 = {} & (1,16,23,27,9,6,25,11,13,12,26,18,7) \\
& (3,19,28,21,22,17,20,8,14,5,4,15,24), \\
\sigma_2 = {} & (3,19,10,25,16,6,21,12,27,17,8,23,14) \\
& (4,7,22,13,28,18,9,24,15,5,20,11,26), \\
\sigma_3 = {} & (1,20,28,2,9,27,23,10,24,14,25,21,15) \\
& (4,18,16,6,13,22,26,11,17,8,5,7,12), \\
\sigma_4 = {} & (3,25,16)(4,13,28)(5,9,18)(6,17,21)(7,26,24) \\
& (8,10,27)(12,19,14)(15,20,22).
\end{aligned}$$

We regard this subgroup as $P\Sigma L(2,27)$.

$N := G^{\mathrm{nil}} = G^{\{3\}} = [G, G]$.

$\mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)}$ has the basis $[E_1] - [E_2]$ such that

$$E_1 = 3V_1 \oplus 3V_3 \oplus 3V_4 \oplus 2V_8,$$
$$E_2 = 3V_6 \oplus 2V_7.$$

Note that $\dim E_1^N = 6$ and $E_2^N = 0$.

$$\sigma_5 = (1, 19, 4, 8, 11, 15, 13, 7, 5, 9, 12, 28, 25, 2)$$
$$(3, 22, 16, 17, 6, 23, 18, 24, 10, 20, 26, 21, 14, 27) \in N$$

$$\mathrm{Ord}(\sigma_5) = 14.$$

The normalizer $N_N(\langle \sigma_5 \rangle) \cong D_{28}$.

By Oliver, $\exists$ a $\mathcal{P}(G)$ -matched, real N-representation pair

$$(U_3, U_4)$$

such that $\dim U_3^N = 1$ and $U_4^N = 0$.
Set $E_3 = \mathrm{ind}_N^G U_3$ and $E_4 = \mathrm{ind}_N^G U_4$.
Then $\dim E_3^G = 1$, $\dim E_3^N = 3$, and $E_4^N = 0$, moreover E_3 has the form

$$E_3 = \mathbb{R} \oplus V_1 \oplus E_f \quad \text{with} \quad E_f^N = 0.$$

§ Representations of $SG(864, 2666)$

Let $G = SG(864, 2666)$. Then the character table of irreducible complex representations of G is as follows.

		1a	3a	2a	6a	2b	2c	4a	4b	4c	4d	3b	3c	6b	3d	3e	6c
	2P	1a	3a	1a	3a	1a	1a	2b	2b	2b	2b	3d	3e	3d	3b	3c	3b
	3P	1a	1a	2a	2a	2b	2c	4c	4d	4a	4b	1a	1a	2c	1a	1a	2c
	5P	1a	3a	2a	6a	2b	2c	4a	4b	4c	4d	3d	3e	6c	3b	3c	6b
X.1		1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2		1	1	1	1	1	1	1	1	1	1	B	B	B	/B	/B	/B
X.3		1	1	1	1	1	1	1	1	1	1	/B	/B	/B	B	B	B
X.4		2	2	2	2	-2	-2	.	.	.	.	-1	-1	1	-1	-1	1
X.5		2	2	2	2	-2	-2	.	.	.	.	-B	-B	B	-/B	-/B	/B

X.6	2	2	2	2	-2	-2	.	.	.	.	-/B	-/B	/B	-B	-B	B
X.7	3	3	3	3	3	3	-1	-1	-1	-1	.	.	.	.	.	.
X.8	3	3	-1	-1	-1	3	1	/A	1	A	.	.	.	.	.	.
X.9	3	3	-1	-1	-1	3	1	A	1	/A	.	.	.	.	.	.
X.10	3	3	-1	-1	-1	3	A	1	/A	1	.	.	.	.	.	.
X.11	3	3	-1	-1	-1	3	/A	1	A	1	.	.	.	.	.	.
X.12	6	6	-2	-2	2	-6	.	.	.	.	.	.	.	.	.	.
X.13	8	-1	8	-1	.	.	.	.	.	.	2	-1	.	2	-1	.
X.14	8	-1	8	-1	.	.	.	.	.	.	C	-/B	.	/C	-B	.
X.15	8	-1	8	-1	.	.	.	.	.	.	/C	-B	.	C	-/B	.
X.16	24	-3	-8	1	.	.	.	.	.	.	.	.	.	.	.	.

A = -1-2*E(4), B = E(3)^2, C = 2*E(3).

The characters of irreducible real G-representations are

```
Y.0 = X.1,
Y.1 = X.2+X.3,
Y.2 = X.4+X.4,
Y.3 =X.5+X.6,
Y.4 = X.7,
Y.5 = X.8+X.9,
Y.6 = X.10+X.11,
Y.7 = X.12,
Y.8 = X.13,
Y.9 = X.14+X.15,
Y.10 = X.16.
```

A basis element of $\mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)}$ is

$$V - W$$

with

$$V = [0, 3, 1, 5, 6, 0, 0, 0, 0, 0, 2],$$
$$W = [0, 0, 0, 0, 0, 2, 2, 4, 2, 2, 0].$$

$$G \supset D_{12}$$

The pair (E_1, E_2) with

$$E_1 = [1, 1, 1, 2, 3, 0, 0, 0, 0, 0, 1],$$
$$E_2 = [0, 0, 0, 0, 0, 1, 1, 2, 1, 1, 0]$$

is a $\mathcal{P}(G)$-matched pair of real type 1.

§ Representations of $SG(864, 4666)$

Let $G = SG(864, 4666)$. The character table of irreducible complex G-representations is as follows.

	1a	3a	2a	2b	6a	2c	4a	4b	4c	4d	3b	3c	6b	3d	3e	6c
2P	1a	3a	1a	1a	3a	1a	2a	2a	2a	2a	3d	3e	3d	3b	3c	3b
3P	1a	1a	2a	2b	2b	2c	4a	4b	4c	4d	1a	1a	2a	1a	1a	2a
5P	1a	3a	2a	2b	6a	2c	4a	4b	4c	4d	3d	3e	6c	3b	3c	6b
X.1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	1	1	1	1	1	1	1	1	1	1	A	A	A	/A	/A	/A
X.3	1	1	1	1	1	1	1	1	1	1	/A	/A	/A	A	A	A
X.4	2	2	-2	2	2	-2	.	.	.	.	-1	-1	1	-1	-1	1
X.5	2	2	-2	2	2	-2	.	.	.	.	-A	-A	A	-/A	-/A	/A

X.6	2	2	-2	2	2	-2	.	.	.	.	-/A	-/A	/A	-A	-A	A
X.7	3	3	3	3	3	3	-1	-1	-1	-1	.	.	.	.	.	.
X.8	3	3	3	-1	-1	-1	-1	3	-1	-1	.	.	.	.	.	.
X.9	3	3	3	-1	-1	-1	-1	-1	3	-1	.	.	.	.	.	.
X.10	3	3	3	-1	-1	-1	-1	-1	-1	3	.	.	.	.	.	.
X.11	3	3	3	-1	-1	-1	3	-1	-1	-1	.	.	.	.	.	.
X.12	6	6	-6	-2	-2	2	.	.	.	.	.	.	.	.	.	.
X.13	8	-1	.	8	-1	.	.	.	.	.	2	-1	.	2	-1	.
X.14	8	-1	.	8	-1	.	.	.	.	.	B	-/A	.	/B	-A	.
X.15	8	-1	.	8	-1	.	.	.	.	.	/B	-A	.	B	-/A	.
X.16	24	-3	.	-8	1	.	.	.	.	.	.	.	.	.	.	.

A = E(3)^2, B = 2*E(3)

Thus the characters of irreducible real G-representations are

```
Y.0 = X.1,
Y.1 = X.2+X.3,
Y.2 = X.4+X.4,
Y.3 = X.5+X.6,
Y.4 = X.7,
Y.5 = X.8,
Y.6 = X.9,
Y.7 = X.10,
Y.8 = X.11,
Y.9 = X.12+X.12,
Y.10 = X.13,
Y.11 = X.14+X.15,
Y.12 = X.16.
```

A basis of $\mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)}$ is

$$V - W$$

with

$$V = [0,3,1,5,6,0,0,0,0,0,0,0,2],$$
$$W = [0,0,0,0,0,2,2,2,2,2,2,2,0].$$

$$G \supset D_{12}$$

The (E_1, E_2) with

$$E_1 = [1,1,1,2,3,0,0,0,0,0,0,0,1],$$
$$E_2 = [0,0,0,0,0,1,1,1,1,1,1,1,0]$$

is a $\mathcal{P}(G)$-matched pair of real type 1.