On the 4-th Johnson homomorphism of the automorphism group of a free group

Takao Satoh

Kyoto University

$\boxed{\text{Introduction}}$

$n \geq 2$

- F_n : free group of rank n
- $\mathrm{Aut}\, F_n$: automorphism group of F_n

- $H := F_n^{\mathrm{ab}} = H_1(F_n, \mathbb{Z}) \cong \mathbb{Z}^{\oplus n}$

$$H \times \mathrm{Aut}\, F_n \to H$$

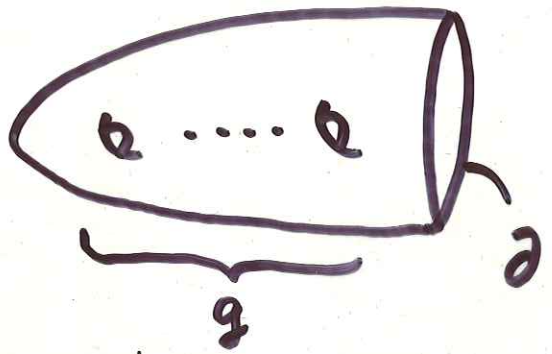
$$(x, \sigma) \mapsto x^\sigma$$

- $\underline{\mathrm{IA}_n := \mathrm{Ker}(\mathrm{Aut}\, F_n \to \mathrm{Aut}(H))}$

IA-automorphism group of F_n

Mapping class group

$g \geq 1$

- $\Sigma_{g,1} :=$ 

- $\mathcal{M}_{g,1} := \mathrm{Diff}^+(\Sigma_{g,1}, \partial)/\text{isotopy}$

$$H := H_1(\Sigma_{g,1}, \mathbb{Z}) \curvearrowleft \mathcal{M}_{g,1}$$

- $\mathcal{I}_{g,1} := \mathrm{Ker}(\mathcal{M}_{g,1} \to \mathrm{Aut}(H))$

Torelli subgroup of $\mathcal{M}_{g,1}$

Theorem. (Dehn, Nielsen) $g \geq 1$

$$\begin{array}{ccc} \iota : \mathcal{M}_{g,1} & \hookrightarrow & \mathrm{Aut}\, \underbrace{F_{2g}}_{\pi_1(\Sigma_{g,1})} \\ \cup & & \cup \\ \mathcal{I}_{g,1} & \hookrightarrow & \mathrm{IA}_{2g} \end{array}$$

Problem For $n \geq 3$

Find a presentation for IA_n.

Johnson 1980

- Johnson homomorphisms

$$\mathrm{IA}_n = \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

$$\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$$

"one by one approximation" of IA_n

$\boxed{\text{Lower central series of } F_n}$

- $\Gamma_n(k)$: lower c. s. of F_n

  $\Gamma_n(1) := F_n$, $\Gamma_n(k) := [\Gamma_n(k-1), F_n]$

$$\begin{array}{ccccccc} \Gamma_n(1) & \supset & \Gamma_n(2) & \supset & \Gamma_n(3) & \supset & \cdots \\ \| & & \| & & & & \\ F_n & & [F_n, F_n] & & & & \end{array}$$

- $\mathcal{L}_n(k) := \Gamma_n(k)/\Gamma_n(k+1)$

Fact. (Magnus, Witt, Hall) $n \geq 2$

(i) $\mathcal{L}_n(k)$: free abelian group

(ii) $\mathrm{rank}_{\mathbb{Z}} \mathcal{L}_n(k) = \dfrac{1}{k} \displaystyle\sum_{d \mid k} \mu(d) n^{k/d}$

μ : Möbius fct.

$\mathcal{L}_n(k)$: "well known"

$\boxed{\text{Johnson filtration of Aut } F_n}$

$F_n/\Gamma_n(k+1) \curvearrowleft \operatorname{Aut} F_n$

$k \geq 1$

- $\underline{\mathcal{A}_n(k) :=}$

$\underline{\operatorname{Ker}(\operatorname{Aut} F_n \to \operatorname{Aut}(F_n/\Gamma_n(k+1))}$

$$\begin{array}{c} \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots \\ \| \\ \mathrm{IA}_n \end{array}$$

Johnson filtration

- S. Andreadakis, 1965

$$\underline{[\mathcal{A}_n(k), \mathcal{A}_n(l)] \subset \mathcal{A}_n(k+l)}$$

- $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$

$\boxed{\text{Johnson homomorphisms}}$

- $k \geq 1$,

$$\tau_k : \mathrm{gr}^k(\mathcal{A}_n) \to \mathrm{Hom}_{\mathbb{Z}}(H, \mathcal{L}_n(k+1))$$

$$\sigma \;\mapsto\; (x \mapsto x^{-1}x^{\sigma})$$

the k-th Johnson hom. of $\mathrm{Aut}\, F_n$

GL(n, Z)-equiv. injective

- Basic problem

$$\mathrm{Im}(\tau_k), \quad \mathrm{Coker}(\tau_k) = ?$$

$\boxed{\text{Some Facts}}$

(1) (Magnus, 1935)

IA_n is generated by

$$K_{ij} : x_i \mapsto {x_j}^{-1} x_i x_j,$$

$$K_{ijk} : x_i \mapsto x_i\,[x_j, x_k], \quad j < k$$

(2) (Andreadakis, 1965)

$$\tau_1 : \mathrm{gr}^1(\mathcal{A}_n) \xrightarrow{\cong} H^* \otimes_{\mathbb{Z}} \Lambda^2 H$$

$H^* := \mathrm{Hom}(H, \mathbb{Z})$

(3) (Cohen-Pakianathan, Farb, Kawazumi)

$$\underset{\substack{\parallel \\ \mathrm{IA}_n^{ab}}}{H_1(\mathrm{IA}_n, \mathbb{Z})} \cong H^* \otimes_{\mathbb{Z}} \Lambda^2 H$$

$[K_{ij}]$, $[K_{ijk}]$: $\mathbb{Z}$-free basis

(4) (S., 2004)

$$\mathrm{Coker}(\tau_2) = S^2 H$$

$$\mathrm{Coker}(\tau_{3,\mathbb{Q}}) = S^3 H_{\mathbb{Q}} \oplus \Lambda^3 H_{\mathbb{Q}}$$

$\tau_{k,\mathbb{Q}} = \tau_k \otimes \mathrm{id}_{\mathbb{Q}}$, $H_{\mathbb{Q}} = H \otimes_{\mathbb{Z}} \mathbb{Q}$

(5) (Morita)

$k \geq 2$

$$\mathrm{Coker}(\tau_{k,\mathbb{Q}}) \supset \underline{S^k H_{\mathbb{Q}}}$$

Morita obst.

Problem $n \geq 3,\ k \geq 4$

$$\mathrm{Coker}(\tau_{k,\mathbb{Q}}) = \ ?$$

Lower central series of IA_n

- $\mathcal{A}'_n(k)$: Lower c. s. of IA_n

$$
\begin{array}{ccccccc}
\mathcal{A}_n(1) & \supset & \mathcal{A}_n(2) & \supset & \mathcal{A}_n(3) & \supset & \cdots \\
\| & & \| & & \cup & & \\
\mathcal{A}'_n(1) & \supset & \mathcal{A}'_n(2) & \supset & \mathcal{A}'_n(3) & \supset & \cdots
\end{array}
$$

Conjecture. For $n \geq 3$, $k \geq 3$,

$$\mathcal{A}_n(k) \underset{?}{=} \mathcal{A}'_n(k)$$

- $\mathrm{gr}^k(\mathcal{A}'_n) := \mathcal{A}'_n(k)/\mathcal{A}'_n(k+1)$

$$\tau'_k : \mathrm{gr}^k(\mathcal{A}'_n) \to H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(k+1)$$

$$\sigma \longmapsto (x \mapsto x^{-1}x^{\sigma})$$

Problem. For $n \geq 3$, $k \geq 4$,

Determine $\mathrm{Coker}(\tau'_k)$.

$\Longrightarrow$

- Using the Magnus generators, we obtain generators of

$$\mathrm{gr}^k(\mathcal{A}'_n).$$

$[K_{ij}, K_{jik}, \cdots, K_{ji}]$

- We can give a lower bound on

$$\mathrm{rank}_{\mathbb{Z}}\, \mathrm{gr}^k(\mathcal{A}_n).$$

- Application to the study of

$$H_2(\mathrm{IA}_n, \mathbb{Z}).$$

- $\mathrm{Im}(\tau_1') = H^* \otimes_{\mathbb{Z}} \Lambda^2 H$
- $\mathrm{Coker}(\tau_2') = S^2 H$
- $\mathrm{Coker}(\tau_{3,\mathbb{Q}}') = S^3 H_{\mathbb{Q}} \oplus \Lambda^3 H_{\mathbb{Q}}$

Facts.

(1) (S. Morita) $k \geq 2$,

$$\mathrm{Coker}(\tau_{k,\mathbb{Q}}') \supset \underline{S^k H_{\mathbb{Q}}}$$

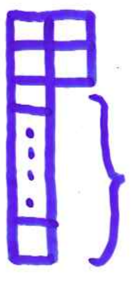
(2) $3 \leq k \leq n, \quad k$: odd,

$$\mathrm{Coker}(\tau_{k,\mathbb{Q}}') \supset \underline{\Lambda^k H_{\mathbb{Q}}}$$

(3) $4 \leq k \leq n-2, \quad k$: even,

$$\mathrm{Coker}(\tau_{k,\mathbb{Q}}') \supset \underline{H_{\mathbb{Q}}^{[2,1^{k-2}]}}$$

$H^{[2,1^{k-2}]} \longleftrightarrow$



$k-2$

Main Theorem

Theorem. (S., 2008) For $n \geq 6$,

$$\mathrm{Coker}(\tau'_{4,\mathbb{Q}}) = S^4 H_{\mathbb{Q}} \oplus H_{\mathbb{Q}}^{[2,1,1]} \oplus H_{\mathbb{Q}}^{[2,2]}$$

new type

$$\dim_{\mathbb{Q}}(\mathrm{Coker}(\tau'_{4,\mathbb{Q}})) = \frac{1}{24}n(n+1)(n+2)(n+3) + \frac{1}{8}n(n+1)(n-1)(n-2) + \frac{1}{12}n^2(n^2-1)$$

Corollary.

$$\mathrm{rank}_{\mathbb{Z}}(\mathrm{gr}^4(\mathcal{A}_n))$$
$$\geq \frac{1}{5}n^2(n^4-1) - \dim_{\mathbb{Q}}(\mathrm{Coker}(\tau'_{4,\mathbb{Q}}))$$

$\boxed{\text{Trace maps}}$

We consider to detect the obstructions by "Trace maps".

- For $k \geq 1$

$$\mathcal{L}_n(k) \hookrightarrow H^{\otimes k}$$

$$[X, Y] \mapsto X \otimes Y - Y \otimes X$$

- $H^* \otimes \mathcal{L}_n(k+1) \to H^* \otimes H^{\otimes(k+1)}$

- **Contraction**

$$\varphi_k^l : H^* \otimes_{\mathbb{Z}} H^{\otimes(k+1)} \to H^{\otimes k}$$

$$x_i^* \otimes x_{j_1} \otimes \cdots \otimes x_{j_{k+1}}$$

l

$$\mapsto x_i^*(x_{j_l}) \cdot x_{j_1} \otimes \cdots \otimes x_{j_{k+1}}$$

$$\Phi_k^l : H^* \otimes \mathcal{L}_n(k+1) \to H^{\otimes k}$$

$\boxed{\text{Morita trace}}$

- $f : H^{\otimes k} \to S^k H$ "abelianization"

$\mathrm{Tr}_{[k]} := f \circ \Phi^1_k$:

$$H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(k+1) \to S^k H$$

Theorem. (Morita) $n \geq 3$, $k \geq 2$,

(1) $\mathrm{Tr}_{[k]}$ is surjective.

(2) $\mathrm{Tr}_{[k]}$ vanishes on $\mathrm{Im}(\tau_k)$.

$$\mathrm{Coker}(\tau_{k,\mathbb{Q}}) \supset S^k H_{\mathbb{Q}}$$

$\boxed{\text{Trace map for } H_{\mathbb{Q}}^{[2,1^{k-2}]}}$

- $g : H^{\otimes k} \to H \otimes_{\mathbb{Z}} \Lambda^{k-1} H$

  $$H_{\mathbb{Q}} \otimes_{\mathbb{Z}} \Lambda^{k-1} H_{\mathbb{Q}} \cong \underline{H_{\mathbb{Q}}^{[2,1^{k-2}]} \oplus \Lambda^k H_{\mathbb{Q}}}$$

- $\mathrm{Tr}_{[2,1^{k-2}]} := g \circ \Phi_k^2$

Theorem. (S., 2004)

For $4 \leq k \leq n-2, \quad k$: even,

(1) $\mathrm{Im}(\mathrm{Tr}^{\mathbb{Q}}_{[2,1^{k-2}]}) = H_{\mathbb{Q}}^{[2,1^{k-2}]}$.

(2) $\mathrm{Tr}_{[2,1^{k-2}]}$ vanishes on $\mathrm{Im}(\tau'_k)$.

$$\mathrm{Coker}(\tau'_{k,\mathbb{Q}}) \supset H_{\mathbb{Q}}^{[2,1^{k-2}]}$$

$$\boxed{\text{Trace map for } H_{\mathbb{Q}}^{[2,2]}}$$

- $H^{[2,2]} \cong \underline{(\Lambda^2 H \otimes_{\mathbb{Z}} \Lambda^2 H)/\sim}$

$$(a \wedge b) \cdot (c \wedge d) := [(a \wedge b) \otimes (c \wedge d)]$$

- $f_i : H^{\otimes 4} \to H^{[2,2]}$, $i = 1, 2$

$$f_i(a \otimes b \otimes c \otimes d) = \begin{cases} (a \wedge c) \cdot (b \wedge d), & i = 1 \\ (a \wedge d) \cdot (b \wedge c), & i = 2 \end{cases}$$

- $\underline{\mathrm{Tr}_{[2,2]} := f_1 \circ \Phi_4^1 - 2(f_2 \circ \Phi_4^1)}$

Theorem. (S., 2008) For $n \geq 6$,

(1) $\mathrm{Tr}_{[2,2]}$ is surjective.

(2) $\mathrm{Tr}_{[2,2]}$ vanishes on $\mathrm{Im}(\tau_4')$.

$$\mathrm{Coker}(\tau'_{4,\mathbb{Q}}) \supset H_{\mathbb{Q}}^{[2,2]}$$