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Borsuk-Ulam type theorems and equilibria in a class of games

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1. A brief description of repeated games with incomplete information

We consider *infinitely repeated two-person, non-zero-sum games of incomplete information on one side on one side*, which were introduced by Aumann, Mashler and Stearns in late 1960's. Further basic results were obtained by S.Sorin in 1983.

There is a finite set K of *states of nature* and two *players* $\mathcal{A}$ and $\mathcal{B}$.

Nature chooses a state $k \in K$ according to a commonly known probability distribution on K. The first player, but not the second player, is informed of nature's choice. The chosen state remains constant throughout the play.

The finite sets of moves for the players, I for $\mathcal{A}$ and J for $\mathcal{B}$, are the same for all states.

The chosen state k, along with the moves of the players, determines the stage payoffs, during the play the second player learns nothing about his payoff, as this could give him information about the state of nature.

Let $m := |I|$ be the number of the first player's actions and $n := |J|$ the number of the second player's actions.

For every state $k \in K$ there are two $m \times n$ matrices A^k and B^k. The i, j entry of A^k is the payoff that the first player receives if the state of nature is k, the first player chooses the action i and the second player chooses the actions j. Likewise the i, j entry of B^k is the payoff that the second player receives if the state of nature is k, the first player chooses the action i and the second player chooses the actions j.

An *equilibrium* of the game is a pair of *strategies* such that

- for every state $k \in K$ there are limits a^k and b^k as the number n of stages go to infinity for the averages summed over the stages up to the stage n of the expected payoffs of Players One and Two, respectively, and
- neither player can obtain a higher limit superior as n goes to infinity for his average payoff summed over the stages up to n (and determined by the initial probability distribution on K) by choosing a different strategy.

More detailed descriptions of equilibria for the games under consideration can be found in [A–M], [So], [Si], [R] and [S–S–T 2].

We divide the problem of equilibrium existence for these games into four levels of difficulty.

• *The first level of difficulty* concerns the conventional game (of standard information): after each stage of play both players are informed of each others' moves and this is the only information the players receive additional to what they knew when the play began. Equilibrium existence for this level was proven by S. Sorin (1983) in the case $|K| \leq 3$, and in general case in [S–S–T 1], 1995.

• For *the second level of difficulty* both players do not know exactly what the other player has done, but at least the perception of the second player is independent of the state. Equilibrium existence for the second level was proven by J. Renault, [R], 2000.

• For *the third level of difficulty* the perception of the second player could be dependent on the state, however the first player has at least some channel with which she can communicate messages that reveal nothing about the state. Equilibrium existence for the third level was established in [S–S–T 2], 2002.

The methods used to prove equilibrium existence for the third level, were applied in [S–S–T 3], 2008, to economic situations broader than that of repeated games of incomplete information, in particular to *principal-agent* situations. A typical principal-agent situation is that of the relationship between the owner of a firm (the principal) and its manager (the agent). The owner employs the manager, both are interested in the success of the firm, but their interests do not coincide and the agent has information on the firm that the owner does not have.

• For *the fourth level of difficulty* there are no assumptions whatsoever concerning the perception of the second player. The question of equilibrium existence for the fourth level remains open. A strategy of a proof of equilibrium existence is described in [S], 2003. It requires a parametrized version of Borsuk–Ulam theorem obtained in [S–S–S–T], 2007, and further advances which still have to be worked out.

2. How do the topological results enter into the play

For the Game with standard information, Aumann, Maschler and Stearns devised strategies (one for each players) which lead to an equilibrium, provided such strategies pairs existed.

They proposed that both players agreed to play at each stage of the game in a certain prescribed manner, known to both of them.

If any of the players violated this agreement, the other one used a punishing strategy that kept the payoff relatively low.

For all of this to work a complicated system of inequalities and equalities needed to be solvable.

It turns out that a solution to this system would exist if we were able to attack the following problem:

Let Δ be an n-simplex in $\mathbf{R}^n$, and let a point $p_0 \in \Delta$ and real functions a and b_v, $v \in \mathbf{R}^n$, on Δ be given, with the functions b_v being convex and continuously depending on v. Let $r : \mathbf{R}^n \to \Delta$ be the nearest point retraction.

Establish conditions under which it is true that

• either b_{p_0} can be separated from a by an affine functional, or

• there exits a finite set $P_0 \subset \mathbf{R}^n$ and an affine functional Φ in $\mathbf{R}^n$ such that $p_0 \in \operatorname{conv}\,(r(P_0))$ and $a \leq \Phi_{|\Delta} \leq b_p$, for all $p \in P_0$.

This geometric problem leads to a setup concerning set–valued functions.

3. Some results on correspondences

By a *correspondence* $F : X \to Y$ we mean a subset of $X \times Y$ such that its projection to X is a proper mapping–that is, it is closed and has compact point-inverses.

For $x \in X, y \in Y$ we consider the sets

$$F(x) := \{y' \in Y : (x, y') \in F\} \ ,$$
$$F^{-1}(y) := \{x' \in X : (x', y) \in F\}$$

In general, we do **not** assume that the set $F(x)$ be non–empty for all $x \in X$.

The set $\operatorname{dom} F := \{x \in X : F(x) \neq \emptyset\}$ is called the *domain* of F, and similarly we define $F(U) = \bigcup_{u \in U} F(u)$, the *image* under F of a subset U of X.

By the *restriction* $F|_S$ of F to a set $S \subset X$ we mean the correspondence $F \cap (S \times Y)$.

We say that F is *acyclic–valued* if, for each $x \in X$, $F(x)$ is non–empty and has trivial reduced Čech homologies with coefficients in $\mathbf{Z}/2$.

Let $(W, \partial W)$ be a relative k-manifold and $F : W \to Y$ a correspondence. We say F has *property* $\mathcal{S}$ (*spanning property*) *for* $(W, \partial W)$ if the homomorphism

$$H_k(F, F_{|\partial W}; \mathbf{Z}/2) \to H_k(W, \partial W; \mathbf{Z}/2)$$

of Čech homology groups induced by the projection to W is an epimorphism. Note that in this case $\mathrm{dom}(F) = W$.

We say $F : \mathbf{R}^n \to Y$ has *property* $\mathcal{S}$ *for an open set* $U \subset \mathbf{R}^n$ if $F_{|\bar{U}}$ has property $\mathcal{S}$ for $(\bar{U}, \partial U)$.

In our setting, X is often going to be a subset of an Euclidean space. Whenever this is the case we may define the following correspondence:

$$cF := \bigcup_{y \in Y} \text{conv } (F^{-1}(y)) \times \{y\}$$

We say that cF is a *level–wise convexification* of F.

Theorem 1. Let U be an open bounded subset of $\mathbf{R}^n$, and let F be an acyclic-valued correspondence defined on the closure of U. If $\dim F(U) < n$, then $c(F_{|\partial U})$ has property $\mathcal{S}$ for U and, consequently, $U \subset \operatorname{dom} c(F_{|\partial U})$.

Thus each point $p_0 \in U$ is in the domain of $c(F_{|\partial U})$, i.e. there exists a finite set $P_0 \subset \partial U$ such that $p_0 \in$ conv (P_0) and $\bigcap_{p \in P_0} F(p) \neq \emptyset$.

• *Application to games.* Theorem 1 allows to give a rather satisfactory answer to the problem on separation of functions. We do not formulate the sufficient condition obtained, the essence of which is that it forces the image of the arising correspondence to have dimension smaller than n and thus makes Theorem 1 applicable. In this way the existence of an equilibrium in the first level of difficulty is established (see [S–S–T 1]).

Applying Theorem 1 and some other properties of correspondences, we prove the following two theorems, Theorem 2 and Theorem 3, needed in the proof of existence of an equilibrium in the third level of difficulty (see [S–S–T 2] and [S–S–T 3]).

For applications to game theory, "*saturated*" correspondences into cubes turn out to be special:

If $F : \Delta(L) \to Y$ is a correspondence and $Y \subset \mathbf{R}^L$, where L is a finite set, then by F^+ we denote the correspondence $\Delta(L) \to Y$ defined by

$$F^+(p) := \{y \in Y : \exists x \in F(p) \text{ such that } x^l \leq y^l$$

$$\text{for all } l \in L \text{ and } x^l = y^l \text{ if } p^l > 0\}.$$

We call F^+ the Y-*saturation* of F and say that F is *saturated* if $F = F^+$. Below, $Y = I^L$ is a cube, with I a non-trivial compact segment in $\mathbf{R}$.

Theorem 2. *Let $\mathbf{C}$ be a family of non-void subsets of a finite set K such that $\bigcup \mathbf{C} = K$. Suppose further there are given a point $p \in \Delta(K)$ and, for every $L \in \mathbf{C}$, a saturated correspondence $F_L : \Delta(L) \to I^L$ with property $\mathcal{S}$ for $\Delta(L)$ and a closed convex subset U_L of I^K containing the point $(b, b, \ldots, b)$. Then there exist a point $y \in \bigcap_{L \in \mathbf{C}} U_L \subset I^K$ and finitely many sets $L_1, \ldots, L_s \in \mathbf{C}$ and points $p_i \in \Delta(L_i) \subset \Delta(K)$, $i = 1, \ldots, s$, such that the following conditions hold:*

$$p \in \text{ conv } \{p_1, \ldots, p_s\} \textit{ and } y^{L_i} \in F_{L_i}(p_i) \textit{ for each } i \ .$$

To warrant property $\mathcal{S}$ of the correspondences F_L we depend on the following

Theorem 3. *Let $F : \Delta \to I^L$ be a convex-valued correspondence and $a : \Delta \to I$ be a lower semi-continuous function such that*

$$a(q) \leq \sup\{y \cdot q : y \in F(p)\} \text{ for all } p, q \in \Delta.$$

With F^+ denoting the I^L–saturation of F, the correspondence $\widetilde{F} : \Delta \to \mathbf{R}^L$ defined by the formula below has property $\mathcal{S}$ for Δ:

$$\widetilde{F}(p) := \{y \in c(F^+)(p) : y \cdot q \geq a(q) \text{ for all } q \in \Delta\}.$$

Above, *convex–valued* means that each set $F(p)$, $p \in \Delta$, is non–empty and convex,

4. Relation to antipodal–type theorems.

From Theorem 1 applied to a single–valued function it follows that when x_0 is a point of a compact set $C \subset \mathbf{R}^n$ and $f : C \to Y$ is a mapping into a space of dimension $n-1$, then in the boundary of C there exists a set C_0 mapped by f into a singleton and containing x_0 in its convex hull.

By Caratheodory's theorem one can always replace C_0 by its subset consisting of $\leq n+1$ points. In general, this number cannot be lowered.

Example. Let f be the natural simplicial map of the barycentric subdivision of an n-simplex Δ onto the join of the barycenter of Δ and of the $(n-2)$-skeleton of Δ. Then the center of Δ cannot be represented as a convex combination of n points that have a common image.

However, in the special case where Y is an $(n-1)$–manifold, a generalization of Borsuk-Ulam theorem given by J. Olędzki implies that C_0 may be taken to consist of 2 points. (Special cases of this were established by K. Sieklucki and K. D. Joshi. The well–known Borsuk–Ulam theorem deals with the case when C is a ball.)

It would be interesting to know less restrictive assumptions under which $n+1$ above could be replaced by a smaller number.

This is related also to estimating the *k–Urysohn diameter* of a compact set C, defined as the infimum of $\sup\{\mathrm{diam}(f^{-1}(y) : y \in f(C)\}$ where f runs over all mappings of C to k–dimensional spaces.

It follows easily from the above, that the $(n-1)$–diameter of a ball in $\mathbf{R}^n$ equals to the "usual" diameter of a regular simplex inscribed into this ball's boundary (apparently this has already been known). When $k \in (n/2, n-2]$ the k-Urysohn diameter of an n–ball remains unknown.

5. A parameterized Borsuk-Ulam theorem

Let W be an k-dimensional compact connected manifold with boundary ∂W (possibly $\partial W = \emptyset$) and let

$$F \subset W \times S^m \times \mathbf{R}^m$$

be compact.

We associate to F its *Borsuk-Ulam correspondence* $G \subset W \times \mathbf{R}^m$ defined by

$$G := \{(w, v) : \exists u \in S^m : (w, u, v), (w, -u, v) \in F\} \; .$$

Note that, if $W = \{pt\}$ and F is the graph of a continuous function $S^m \to \mathbf{R}^m$, then the Borsuk-Ulam theorem states that G is non-empty, whence the chosen name.

Theorem 4. Suppose that the correspondence $F : W \times S^m \to \mathbf{R}^m$ has property $\mathcal{S}$ for $(W, \partial W) \times S^m$. Then the correspondence $G : W \to \mathbf{R}^m$ has property $\mathcal{S}$ for $(W, \partial W)$. Consequently, G is mapped onto W by the projection $W \times \mathbf{R}^m \to W$.

Notation. Below, by H we denote the Čech homology functor with coefficients in $\mathbf{Z}/2$.

An equivalent statement:

Suppose that the homomorphism

$$p_* : H_{k+m}(F, F_{|\partial W \times S^m}) \to H_{k+m}((W, \partial W) \times S^m) \ ,$$

where $p : (F, F_{|\partial W \times S^m}) \to (W, \partial W) \times S^m$ is the map induced by the projection $W \times S^m \times \mathbf{R}^m \to W \times S^m$, is an epimorphism. Then

$$q_* : H_k(G, G_{|\partial W}) \to H_k(W, \partial W) \ ,$$

where $q : (G, G_{|\partial W}) \to (W, \partial W))$ is the map induced by the projection $W \times \mathbf{R}^m \to W$, is an epimorphism as well.

Remark. Theorem 4 is the parametrized Borsuk-Ulam theorem. Loosely speaking, it asserts the following: *if one has a family of Borsuk-Ulam problems parametrized by a manifold W which depend "continuously" on the points in W, then the solutions to the Borsuk-Ulam problem can be chosen to depend "continuously" on the points in W as well.*

"Continuity" here is measured by the fact that one can span a Čech-homology class through the set which projects down to the fundamental class of W.

Sketch of the proof of Theorem 4

Let $[(W, \partial W) \times S^m]$ and $[W, \partial W]$ be fundamental classes of the manifolds $(W, \partial W) \times S^m$ and $(W, \partial W)$, respectively.

Note that to prove the theorem it suffices to define a map

$$H_{k+m}(F, F_{|\partial W \times S^m})) \to H_k(G, G_{|\partial W}), \quad f \mapsto g_f$$

such that $q_*(g_f) = [(W, \partial W)]$ provided $p_*(f) = [(W, \partial W) \times S^m]$.

To define this map, we need to establish *a relative squaring construction* in Čech homology.

Let (X, A) be a pair of compacta. On $X \times X$, we have the involution τ with $\tau(x, y) = (y, x)$. Consider the quotient pair

$$SP(X, A) := ((X \times X)/\tau, (X \times A \cup A \times X \cup D)/\tau) \ ,$$

where $D := \{(x, x) \mid x \in X\}$ is the diagonal.

Then there is a natural homomorphism, called *invariant homology squaring*

$$Sq : H_k(X, A) \to H_{2k}(SP(X, A)) \ ,$$

with the following properties:

(1) Let M be a manifold with the boundary ∂M and let $[M, \partial M] \in H_{\dim M}(M, \partial M)$ be the fundamental class of $(M, \partial M)$. Then the fundamental class $[SP(M, \partial M)]$ of the relative manifold $SP(M, \partial M)$ is equal to $Sq[M, \partial M]$.

(2) Let (X, A) and (Y, B) be pairs of compacta such that $(X, A) \subset (Y, B)$. Then

$$Sq(i_*(x)) = j_*(Sq(x))$$

for each $x \in H_*(X, A)$, where $i : (X, A) \hookrightarrow (Y, B)$ and $j : SP(X, A) \hookrightarrow SP(Y, B)$ denote the inclusions.

(3) Let (X, A) and (Y, B) be arbitrary pairs of compact subsets of a manifold M, and let $x \in H_k(X, A)$ and $y \in H_l(Y, B)$. Then

$$Sq(x \bullet y) = Sq(x) \bullet Sq(y),$$

where $\bullet$ denotes the intersection of homology classes. (Let us recall, that

$$x \bullet y \in H_{k+l-\dim M}(X \cap Y, X \cap B \cup A \cap Y)\ .)$$

Using the invariant homology squaring, we describe for each $f \in H_{k+m}(F, F_{|\partial W \times S^m})$ the required element $g_f \in H_k(G, G_{|\partial W})$ as follows.

We define the "antidiagonal"

$$\Delta := \{[w, u, v, w, -u, v]\} \subset SP(W \times S^m \times \mathbf{R}^m) \ .$$

Observe that

$$\Delta \cong W \times \mathbf{R}P^m \times \mathbf{R}^m \ .$$

By $[\Delta, \partial\Delta] \in H_{3k+2m}(\Delta, \partial\Delta)$ we denote the fundamental class of the manifold $(\Delta, \partial\Delta)$ (we use locally finite homology).

Let $(F', \partial F') := SP(F, F_{|\partial W \times S^m}) \cap (\Delta, \partial\Delta)$. We define

$$g_f := (\pi_G)_* \left(Sq(f) \bullet [\Delta, \partial\Delta] \right) \ ,$$

where $\pi_G : (F', \partial F') \to (G, G_{|\partial W})$ is induced by the projection $\Delta \cong W \times \mathbf{R}P^m \times \mathbf{R}^m \to W \times \mathbf{R}^m$.

One can observe that

$$q_*(g_f) = (\pi_W \circ j_{F'})_*(Sq(f) \bullet [\Delta, \partial\Delta]) \ ,$$

where $j_{F'} : (F', \partial F') \to (\Delta, \partial\Delta)$ is the inclusion, and $\pi_W : (\Delta, \partial\Delta) \cong (W, \partial W) \times \mathbf{R}P^m \times \mathbf{R}^m \to (W, \partial W)$ denotes the projection.

Thus, to conclude the proof it suffice to show that

(a) $\qquad (\pi_W \circ j_{F'})_*(Sq(f) \bullet [\Delta, \partial\Delta]) = [W, \partial W]$

provided $p_*(f) = [(W, \partial W) \times S^m]$.

To prove (a), we first show that

(b) $\qquad Sq(f) = [E, \partial E]$ in $H_{2(k+m)}(\widehat{W}, \widehat{\partial W})$,

where $(\widehat{W}, \widehat{\partial W}) := SP((W, \partial W) \times S^m \times \mathbf{R}^m)$, E is the subset of $\widehat{W}$ consisting of all points

$$[w, (u, u_{m+1}), u, w', (u', u'_{m+1}), u']$$

such that $u_{m+1}, u'_{m+1} \in S^m$, and $\partial E := E \cap \widehat{\partial W}$.

Notation. If $(X_i, A_i) \subset (Y, B)$, $i = 1, 2$, and $x_i \in H_*(X_i, A_i)$, then we say that x_1 and x_2 *are equal in* $H_*(Y, B)$ if there images under the homeomorphisms of homology groups induced by the corresponding inclusions are equal.

By (b), in $H_k(\Delta, \partial\Delta)$ we have

$$Sq(f) \bullet [\Delta, \partial\Delta] = [E, \partial E] \bullet [\Delta, \partial\Delta] \ .$$

Since E and Δ intersect transversally it follows that

$$[E, \partial E] \bullet [\Delta, \partial\Delta] = [E \cap \Delta, \partial E \cap \partial\Delta] \ .$$

Next, because $\pi_W|(E \cap \Delta, \partial E \cap \partial\Delta)$ is a homeomorphism onto $(W, \partial W)$, it follows that

$$(\pi_w \circ j_{E\cap\Delta})_*([E \cap \Delta, \partial E \cap \partial\Delta]) = [W, \partial W] \ ,$$

where $j_{E\cap\Delta} : (E \cap \Delta, \partial E \cap \partial\Delta) \hookrightarrow (\Delta, \partial\Delta)$ is the inclusion.

Consequently,

$$(\pi_w \circ j_{F'})_*(Sq(f) \bullet [\Delta, \partial\Delta]) = [W, \partial W]$$

since in $H_k(\Delta, \partial\Delta)$ we have

$$Sq(f) \bullet [\Delta, \partial\Delta] = [E \cap \Delta, \partial E \cap \partial\Delta] \ .$$

This completes the proof of Theorem 4.

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