Preliminary Report

Homeomorphism and Diffeomorphism Groups of Non-Compact Manifolds with the Whitney Topology

Tatsuhiko Yagasaki (Kyoto Institute of Technology)

Joint Work with **T. Banakh, K. Sakai, K. Mine**

The 35th Symposium on Transformation Groups

Okayama, Nov. 13, 2008

Contents :

Ch 1. Criterion that Top group $\approx_{loc} \mathbb{R}^\infty \times \ell^2$

G : Top group (Top linear space)

Fundamental Problem

Classification of Local/Global Top type of G

" When is $G \approx_{loc} L$ (Linear space) ? "

§1. Known Results

(1) $\mathbb{R}^n$ case — A. Gleason, D. Montgomery & L. Zippin

(5 th Problem of D. Hilbert)

G : Locally compact, Separable ANR

$\Longrightarrow$ G : Lie group $\quad \therefore\ G \approx_{loc} \mathbb{R}^n$

(2) ℓ_2 case — T. Dobrowolski & H. Toruńczyk

G : Non-locally compact, Completely metrizable,

Separable, ANR $\Longrightarrow$ $G \approx_{loc} \ell_2$

(3) Fréchet space = Completely metrizable, locally convex topological linear space

Separable Fréchet space $\approx \ell_2$ $\qquad \prod^{\infty}\mathbb{R} \approx \ell_2$

(4) LF-space

$=$ Direct limit of increasing sequence of Fréchet spaces
in Category of locally convex topological linear spaces

Example : $\mathbb{R}^\infty =$ Direct limit of $\mathbb{R}^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3 \subset \cdots$

P. Mankiewicz :

Infinite-dim separable LF-space $\approx$ $\mathbb{R}^\infty$, ℓ_2 or $\mathbb{R}^\infty \times \ell_2$

T. Banakh, K. Mine, K. Sakai, T. Yagasaki

Criterion that $G \approx_{(loc)} \mathbb{R}^\infty \times \ell_2$

§2. Small box products

§3. Tower of subgroups

§4. Topologically complemented subgroups

§2. Small box products

Definition 2.1.

(1) $(X_n)_{n\geq 1}$: Sequence of Topological spaces

Box product $\quad \square_n X_n = \prod_n X_n \qquad$ (Box topology)

Basic open set $\quad \prod_n U_n \qquad (U_n \subset X_n$: open$)$

(2) $(X_n, *_n)_{n\geq 1}$: Sequence of pointed spaces

Small box product $\quad \boxdot_n X_n \ \subset\ \square_n X_n$

$$(x_1, x_2, \cdots, x_n, *, *, \cdots)$$

$\boxdot^\omega \ell_2 \approx \mathbb{R}^\infty \times \ell_2$

§3. Tower of subgroups

G : Top group ($e \in G$: the unit element)

Definition 3.2. Tower of closed subgroups of G

= Sequence $(G_n)_n$ of closed subgroups of G such that

$$G_1 \subset G_2 \subset G_3 \subset \cdots, \quad G = \bigcup_n G_n$$

Small box product $\boxdot_n G_n \ni \boldsymbol{e} = (e, e, \dots)$

Multiplication map $p : \boxdot_n G_n \to G$

$$p(x_1, x_2, \dots, x_n, e, e, \cdots) = x_1 x_2 \cdots x_n$$

§4. Topologically complemented subgroups

G : Top group $\qquad H \subset G$: Closed subgroup

$\pi : G \to G/H : \quad \pi(g) = gH$

Definition 4.3.

(1) H : locally top complemented (loc TC) in G

$\iff$ π has a local section

$\iff$ π : a principal H-bundle

(2) H : top complemented (TC) in G

$\iff$ π has a section $\iff$ π : a trivial H-bundle

Definition 4.4. $(G_n)_n$: Tower of closed subgroups of G

(G_n) : (loc) TC $\iff$ $\forall\, n \quad G_n$: (loc) TC in G_{n+1}

§5. Criterion that Top group $\approx_{(loc)} \mathbb{R}^\infty \times \ell_2$

G : Top group $\quad (G_n)_n$: Tower of closed subgroups of G

$(*_1)\quad p : \boxdot_n G_n \to G$: open

$(*_2)_{(loc)}\quad (G_n)_n$: (loc) TC

$(*_3)_{(loc)}\quad G_n/G_{n-1} \approx_{(loc)} \ell_2 \quad (n \geq 1)$

Theorem 5.1. $G \approx_{(loc)} \mathbb{R}^\infty \times \ell_2$

Proof.

$$\mathbb{R}^\infty \times \ell_2 \approx \boxdot_n \ell_2 \approx_{(loc)} \boxdot_n (G_n/G_{n-1}) \xrightarrow[\approx_{(loc)}]{qs} G$$

$$s=(s_n)_n : \boxdot_n (G_n/G_{n-1}) \to \boxdot_n G_n, \quad q : \boxdot_n G_n \to G$$

$$G_n \underset{s_n}{\overset{\pi_n}{\rightleftarrows}} G_n/G_{n-1} \qquad q(x_0, \ldots, x_n, e, e, \cdots) = x_n \cdots x_0$$

□

$H \subset G,\ \ H_n \subset G_n \ \ (n \geq 1)$: Identity components

Theorem 5.2. $G : (*_1), (*_2)_{loc}, (*_3)_{loc}$

(1) $H \subset G$: Open normal subgroup

G/H : Discrete topology $\quad \therefore \quad G \approx H \times G/H$

(2) $H_n \simeq * \ \ (n \geq 1) \implies H \approx \mathbb{R}^\infty \times \ell_2$

(3) $H_1 \simeq L$ (a locally compact polyhedron)

$\big(G_n/G_{n-1}\big)_0 \simeq * \ \ (n \geq 2)$

$\implies H \approx L \times \mathbb{R}^\infty \times \ell_2$

$\big(G_n/G_{n-1}\big)_0$: Connected Component of G_{n-1} in G_n/G_{n-1}

Ch 2. Applications to Homeo / Diffeo groups with Whitney topology

§1. Homeo groups of Top manifolds

M : Top n-manifold $\mathcal{H}(M)$: Homeo group of M

§1.1. Whitney topology on $\mathcal{H}(M)$

(1) Basic fundamental neighborhood system :

$f \in \mathcal{H}(M) \quad \mathcal{U} \in \mathrm{cov}(M)$

$$\mathcal{U}(f) = \left\{ g \in \mathcal{H}(M) \mid (f, g) \prec \mathcal{U} \right\}$$

$\circ$ $(f, g) \prec \mathcal{U} \iff \forall\, x \in M \;\; \exists\, U \in \mathcal{U}$ with $f(x), g(x) \in U$

(2) $\mathcal{H}_0(M)$: Identity component of $\mathcal{H}(M)$

$$\mathcal{H}_0(M) \subset \mathcal{H}_c(M)$$

(3) Mapping class group $\mathcal{M}_c(M) = \mathcal{H}_c(M)/\mathcal{H}_0(M)$

M : Non-compact connected n-manifold $\qquad \partial M = \emptyset$

§1.2. $n = 1 \qquad (\mathcal{H}(\mathbb{R}), \mathcal{H}_c(\mathbb{R})) \approx (\square^\omega \ell_2, \boxdot^\omega \ell_2)$

§1.3. $n = 2$

(1) $(\mathcal{H}(M), \mathcal{H}_c(M)) \approx_{loc} (\square^\omega \ell_2, \boxdot^\omega \ell_2))$

(2) $\mathcal{H}_0(M) \approx \mathbb{R}^\infty \times \ell_2 \quad (\simeq *) \qquad \mathcal{M}_c(M)$: Discrete Top

(3) $\mathcal{H}_c(M) \approx \mathcal{H}_0(M) \times \mathcal{M}_c(M) \approx \begin{cases} \mathbb{R}^\infty \times \ell_2 & (\#) \\ \mathbb{R}^\infty \times \ell_2 \times \mathbb{Z} & \text{other cases} \end{cases}$

Lemma 1.1. The following conditions are equivalent:

$(\#_1)$ $\mathcal{M}_c(M)$: Trivial $\qquad (\#_2)$ $\mathcal{M}_c(M)$: Torsion group

$(\#_3)$ $M \approx X \setminus C$, where $\qquad X = \mathbb{A}, \mathbb{D}$ or $\mathbb{M}$

C : Non-empty compact subset of a boundary circle of X

Comparison with Compact-Open Topology

Remark 1.1. :

(1) $\mathcal{H}_0(M) \approx \mathbb{R}^\infty \times \ell_2$ $\qquad$ $\mathcal{H}_c(M)_0 = \mathcal{H}_0(M)$

(2) $\mathcal{H}_0(M)_{co} \approx \begin{cases} \mathbb{S}^1 \times \ell_2 & \text{if } M \approx \mathbb{R}^2, \mathbb{S}^1 \times \mathbb{R} \text{ or } \mathbb{M} \setminus \partial\mathbb{M} \\ \ell_2 & \text{all other cases} \end{cases}$

$$\mathcal{H}_0(M)_{co} \overset{\simeq}{\supset} \mathcal{H}_c(M)_0$$

§1.4. $n \geq 3$

(1) $\mathcal{H}_c(M)$: locally contractible

§1.2. (Weak) Extension Theorem for Embeddings

M : n-MFD $\qquad C \subset D \subset M$: Compact $\qquad C \subset \mathrm{int}_M D$

$$\mathcal{H}(M, M - D) = \{h \in \mathcal{H}(M) : h = id \text{ on } M - D\}$$

$$\mathcal{E}^\star(C, M) = \{h|_C : h \in \mathcal{H}(M)\} \qquad \mathcal{E}^\star(D, M) = \ldots$$

Theorem 2.3. (Edwards - Kirby) $\quad i_D \in \mathcal{V} \subset \mathcal{E}^\star(D, M)$

$$\begin{array}{ccc} & & \mathcal{H}(M, M - D) \\ & \overset{\exists\,\varphi}{\nearrow} & \downarrow \text{res.} \\ i_D \in {}^\exists\mathcal{V} & \xrightarrow[\text{res.}]{} & \mathcal{E}^\star(C, M) \end{array} \qquad \varphi(f)|_C = f|_C \quad (f \in \mathcal{V})$$

Theorem 2.4. (Luke - Mason, Yagasaki, et. al)

M : 2-MFD $\qquad C$: Compact subpolyhedron

$$\begin{array}{ccc} & & \mathcal{H}(M, M - D) \\ & \overset{\exists\,\varphi}{\nearrow} & \downarrow \text{res.} \\ i_C \in {}^\exists\mathcal{V} & \hookrightarrow & \mathcal{E}^\star(C, M) \approx_\ell l_2 \end{array} \qquad \varphi(f)|_C = f \quad (f \in \mathcal{V})$$

§2. Diffeo groups of Smooth manifolds

- M : Non-compact connected C^∞ n-manifolds $\quad \partial M = \emptyset$

  $\mathcal{D}(M)$: Diffeo group of M $\quad$ (Whitney C^∞-topology)

  $\mathcal{D}_0(M) \subset \mathcal{D}(M)$: Identity component

  $\mathcal{D}_c(M) \subset \mathcal{D}(M)$: Compact support $\quad \mathcal{D}_0(M) \subset \mathcal{D}_c(M)$

  $\mathcal{M}_c^\infty(M) = \mathcal{D}_c(M)/\mathcal{D}_0(M)$

  $\mathcal{D}(M, X) = \{h \in \mathcal{D}(M) : h|_X = \mathrm{id}\}$

  $\mathcal{D}_0(M, X) \subset \mathcal{D}(M, X)$: Identity component

- $K \subset L$: subsets of M

  $\mathcal{E}_K^{\infty,\star}(L, M) =$ Space of C^∞-embeddings $f : L \longrightarrow M$ such that

  $f|_K = \mathrm{id}_K, \quad f = g|L$ for some $g \in \mathcal{D}(M)$

- Extension Theorem for C^∞-embeddings (Cerf, Palais, et al.)

Theorem 2.5.

(1) $(\mathcal{D}(M), \mathcal{D}_c(M)) \approx_{loc} (\square^\omega \ell_2, \boxdot^\omega \ell_2)$

$\mathcal{D}_c(M) \approx_{loc} \mathbb{R}^\infty \times \ell_2$

(2) $\mathcal{D}_0(M) \lhd \mathcal{D}_c(M)$: clopen

$\mathcal{D}_c(M) \approx \mathcal{D}_0(M) \times \mathcal{M}_c^\infty(M)^{\text{discrete top}}$

(3) $(M_i)_{i \geq 1}$: Sequence of compact n-submanifolds of M s.t.

$$M_i \subset \operatorname{Int} M_{i+1} \ \ (i \geq 1), \quad M = \bigcup_i M_i$$

$K_i := M \setminus \operatorname{Int} M_i \ \ (i \geq 1)$

(i) $\mathcal{D}_0(M, K_i) \simeq * \ \ (i \geq 1) \implies \mathcal{D}_0(M) \approx \mathbb{R}^\infty \times \ell_2$

(ii) $\mathcal{E}^{\infty,\star}_{K_{i+1}}(K_i, M)_0 \simeq * \ \ (i \geq 1)$

$$\implies \mathcal{D}_0(M) \approx \mathcal{D}_0(M, K_1) \times (\mathbb{R}^\infty \times \ell_2).$$

Corollary 2.1.

(1) $n = 1, 2 :\quad \mathcal{D}_0(M) \approx \mathbb{R}^\infty \times \ell_2$

(2) $n = 3 :\ M :$ orientable, irreducible $\Longrightarrow\ \mathcal{D}_0(M) \approx \mathbb{R}^\infty \times \ell_2$

- M : irreducible $\Longleftrightarrow$ any 2-sphere in M bounds a 3-ball
- Example : $\mathbb{R}^3$ or Whitehead contractible 3-manifold

(3) $\forall\, n :\quad M = \operatorname{Int} X$

X : Compact connected C^∞ n-manifold **with boundary**

$\Longrightarrow\ \mathcal{D}_0(M) \approx \mathcal{D}_0(X, \partial X) \times \mathbb{R}^\infty \times \ell_2$

- $\mathcal{D}_0(X, \partial X) \simeq L\ \Longrightarrow\ \mathcal{D}_0(M) \approx L \times \mathbb{R}^\infty \times \ell_2$