§1 ~ §3

1.3

(1) $f : (C, p) \longrightarrow (D, q)$ $\quad\varepsilon$ chain map $\qquad df \underset{\varepsilon}{\sim} fd$

$f' : (D, q) \longrightarrow (E, r)$ $\quad\varepsilon'$ chain map

$f'f : (C, p) \longrightarrow (E, r)$

各次元で $\varepsilon + \varepsilon'$ morphismである。

$$d(f'f) = \underset{\underset{(f'd\ )}{\leq \varepsilon'}}{(df')}\overset{\varepsilon}{f} \underset{\varepsilon+\varepsilon'}{\sim} (f'd)f = f'(df) \underset{\varepsilon'+\varepsilon}{\sim} f'(fd)$$

$$\therefore\ d(f'f) \underset{\varepsilon+\varepsilon'}{\sim} (f'f)d$$

$\therefore$ $f'f$ は $\varepsilon + \varepsilon'$ chain map

(2) $f : (C, p) \longrightarrow (D, q)$ $\quad\varepsilon$ isomorphism $\qquad$ 逆 : g

$f' : (D, q) \rightarrow (E, r)$ $\quad\varepsilon'$ 〃 $\qquad$: g'

$f'f : (C, p) \rightarrow (E, r)$: $\varepsilon + \varepsilon'$ chain map

$g g' : (E, r) \rightarrow (C, p)$: $\varepsilon + \varepsilon'$ 〃 〃

$$(gg')(f'f) \underset{2\varepsilon+2\varepsilon'}{\sim} g\, q\, f \underset{2\varepsilon}{\sim} gf \underset{2\varepsilon}{\sim} p$$

$$\therefore\ (gg')(f'f) \underset{2(\varepsilon+\varepsilon')}{\sim} p$$

$$(f'f)(gg') \underset{2(\varepsilon+\varepsilon')}{\sim} r$$

$\therefore$ $\varepsilon + \varepsilon'$ isomorphism.

(3) $f : (C, p) \longrightarrow (D, q)$ $\quad\varepsilon$ chain eq. $\qquad$ 逆 g

$f' : (D, q) \rightarrow (E, r)$ $\quad\varepsilon'$ 〃 〃. $\qquad g'$

$h : gf \simeq p$, $\quad k : fg \simeq q$

$dh + hd \underset{2\varepsilon}{\sim} p - gf \qquad dk + kd \underset{2\varepsilon}{\sim} q - fg$

$h' : g'f' \simeq \ell$ $k' : f'g' \simeq r$

$dh' + h'd \underset{2\varepsilon}{\sim} \ell - g'f'$ $dk' + k'd \underset{2\varepsilon'}{\sim} r - f'g'$.

$$d(f'kg' + k') + (f'kg' + k')d$$

$$\overset{2\varepsilon'}{}$$

$$= \underbrace{df'}_{\varepsilon'} \underset{\varepsilon}{\boxed{k}} \underset{\varepsilon'}{g'} + dk' + \underset{\varepsilon'}{f'} \underset{\varepsilon}{k} \underbrace{g'd}_{\varepsilon'} + k'd$$

$$\underset{\varepsilon+2\varepsilon'}{\sim} f'(\underline{dk + kd})g' + (r - f'g')$$

$$\underset{2\varepsilon+2\varepsilon'}{\sim} f'(\ell - fg)g' + (r - f'g')$$

$$\underset{2\varepsilon+2\varepsilon'}{\sim} f'g' - f'fgg' + r - f'g'$$

㊟

*. ε chain map の def を $df \underset{2\varepsilon}{\sim} fd$ にすると. ここで. もんだいが

おこる

$$df'kg' \underset{\varepsilon+3\varepsilon'}{\sim} f'dkg'$$

1.4.

$f : (C, p) \longrightarrow (D, q)$ ε chain map

① $\mathcal{C}(f)$: ε contractible

($\Gamma : 0 \simeq q \oplus p : \mathcal{C}(f) \longrightarrow \mathcal{C}(f)$ ε chain contraction)

$\Rightarrow$ f is a 2ε chain equivalence

(∵)

$$\Gamma = \begin{pmatrix} k & * \\ (-)^r g & h \end{pmatrix} : \mathcal{C}(f)_r = (D_r, q_r) \oplus (C_{r-1}, p_{r-1})$$

$$\longrightarrow \mathcal{C}(f)_{r+1} = (D_{r+1}, q_{r+1}) \oplus (C_r, p_r)$$

とおく.

<u>def</u> より. k, g, h は ε morphisms (*) → 次ページをみよ!!

$$d\Gamma + \Gamma d : \mathcal{C}(f)_r \longrightarrow \mathcal{C}(f)_r$$

$$= \overset{r+1 \to r}{\begin{pmatrix} d & (-)^r f \\ 0 & d \end{pmatrix}} \overset{r \to r+1}{\begin{pmatrix} k & * \\ (-)^r g & h \end{pmatrix}} + \overset{r-1 \to r}{\begin{pmatrix} k & * \\ (-)^{r-1} g & h \end{pmatrix}} \overset{r \to r-1}{\begin{pmatrix} d & (-)^{r-1} f \\ 0 & d \end{pmatrix}}$$

$$= \begin{pmatrix} dk + fg & \cdots \\ (-)^r dg & dh \end{pmatrix} + \begin{pmatrix} kd & \cdots \\ (-)^{r-1} gd & gf + hd \end{pmatrix}$$

$$= \begin{pmatrix} dk + kd + fg & \cdots \\ (-)^r dg + (-)^{r-1} gd & dh + hd + gf \end{pmatrix}$$

仮定より. $\underset{2\varepsilon}{\sim} \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix}$

$\therefore$ $dk + kd + fg \underset{2\varepsilon}{\sim} q$

$\therefore$ $dk + kd \underset{2\varepsilon}{\sim} q - fg$

同様に $dh + hd \underset{2\varepsilon}{\sim} p - gf$

よって. $(-)^r(dg - gd) \sim_{2\varepsilon} 0$.

＊ その前に忘れてならないのが. ちゃんと ε morphisms になっているかどうか？

$$\begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} k & \cdots \\ (-)^r g & h \end{pmatrix} \sim_\varepsilon \begin{pmatrix} k & \cdots \\ (-)^r g & h \end{pmatrix}$$

$$\parallel$$

$$\begin{pmatrix} qk & \cdots \\ (-)^r pg & ph \end{pmatrix}$$

$\therefore\ qk \sim_\varepsilon k$

$(-)^r pg \sim_{2\varepsilon} (-)^r g \quad \therefore\ pg \sim g$

$ph \sim h$

$$\begin{pmatrix} k & \cdots \\ (-)^r g & h \end{pmatrix}\begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix} \sim_\varepsilon \begin{pmatrix} k & \cdots \\ (-)^r g & h \end{pmatrix}$$

$$\parallel$$

$$\begin{pmatrix} kq & \cdots \\ (-)^r gq & hp \end{pmatrix}$$

$\therefore\ kq \sim_\varepsilon k,\quad gq \sim_\varepsilon g,\quad hp \sim_\varepsilon h$.

よって k, g, h いずれも ε morphism.

しかし. g は 2ε chain map [∵ $dg \sim_{2\varepsilon} gd$]

② 逆に. $f : (C, p) \rightarrow (D, q)$ ε chain aquivalence

$$g \; (\leftarrow)$$

$$\begin{cases} \hbar : gf \simeq p \\ \bar{k} : fg \simeq q \end{cases} \quad \varepsilon \text{ chain htpies} \qquad \begin{aligned} d\hbar + \hbar d &\underset{2\varepsilon}{\sim} p - gf \\ d\bar{k} + \bar{k}d &\underset{2\varepsilon}{\sim} q - fg \end{aligned}$$

$$\Gamma = \begin{pmatrix} k + (fh - \bar{k}f)g & (-)^r(fh - \bar{k}f)\hbar \\ (-)^r g & \hbar \end{pmatrix}$$: 各次元で $\boxed{3\varepsilon}$ morphism

$$: \mathcal{C}(f)_r \longrightarrow \mathcal{C}(f)_{r+1}$$

$$d\Gamma + \Gamma d$$

$$= \begin{pmatrix} d & (-)^r f \\ 0 & d \end{pmatrix} \begin{pmatrix} k + (fh - \bar{k}f)g & (-)^r (fh - \bar{k}f)h \\ (-)^r g & h \end{pmatrix}$$

$$+ \begin{pmatrix} k + (fh - \bar{k}f)g & (-)^{r-1}(fh - \bar{k}f)h \\ (-)^{r-1} g & h \end{pmatrix} \begin{pmatrix} d & (-)^{r-1} f \\ 0 & d \end{pmatrix}$$

$$= \begin{pmatrix} \underline{dk} + d(f\hbar - \bar{k}f)g + \underline{fg} & (-)^r d(fh - \bar{k}f)h + (-)^r fh \\ (-)^r dg & d\hbar \end{pmatrix}$$

$$+ \begin{pmatrix} \underline{kd} + (fh - \bar{k}f)gd & (-)^{r-1}kf + (-)^{r-1}(fh - \bar{k}f)gf + (-)^{r-1}(fh - \bar{k}f)hd \\ (-)^{r-1} gd & gf + hd \end{pmatrix}$$

(1,1)成分 $= \underbrace{d\hbar + \hbar d + fg}_{2\varepsilon} + (\overset{\varepsilon}{\underline{df}}\overset{\varepsilon}{\hbar}\overset{\varepsilon}{g} + f\hbar\overset{\varepsilon}{\underline{\underline{gd}}}) - (\overset{2\varepsilon}{d\bar{k}}\overset{\varepsilon}{f}\overset{\varepsilon}{g} + \bar{k}f\overset{\varepsilon}{\underline{\underline{gd}}})$

$$\underset{4\varepsilon}{\sim} q + (fd\hbar g + f\hbar dg) - (d\bar{k}fg + \bar{k}\underline{\underline{fd}}g)$$

$$\underset{4\varepsilon}{\sim} q + \overset{\varepsilon}{f}(\overset{2\varepsilon}{\overbrace{d\hbar + \hbar d}})\overset{\varepsilon}{g} - (\overset{2\varepsilon}{\overbrace{d\bar{k} + \bar{k}d}})\overset{\varepsilon}{f}\overset{\varepsilon}{g}$$

$$\underset{4\varepsilon}{\sim} q + f(p - gf)g - (q - fg)fg$$

$$= q + fpg - fgfg - qfg + fgfg$$

$$\underset{4\varepsilon}{\sim} q + fg - fg \sim q .$$

(1,2)成分 $= (-)^r\{ d(f\hbar - kf)\hbar + f\hbar - kf - (f\hbar - kf)gf - (f\hbar - kf)hd \}$

$= (-)^r\{ df\hbar\hbar - dkf\hbar + f\hbar - kf - f\hbar gf + kfgf - f\hbar\hbar d + kfhd \}$

$\sim (-)^r\{ fd\hbar\hbar - dkf\hbar + \underline{f\hbar p} - \underset{\smile}{kfp} - \underline{f\hbar gf} + \underset{\smile}{kfgf} \underline{- f\hbar\hbar d} + \underset{\smile}{kf\hbar d} \}$

$= (-)^r\{ fd\hbar\hbar - dkf\hbar + f\hbar(\underbrace{p - gf - \hbar d}_{\overset{\wr}{d\hbar}}) - kf(\underbrace{p - gf - \hbar d}_{\overset{\wr}{d\hbar}}) \}.$

$\sim (-)^r[\underline{fd\hbar\hbar} - dkf\hbar + \underline{f\hbar d\hbar} - k\textcircled{fd}\hbar]$

$\sim (-)^r\{ f(d\hbar + \hbar d)\hbar - dkf\hbar - kdf\hbar \}$

$= (-)^r\{ f(d\hbar + \hbar d)\hbar - (dk + kd)f\hbar \}$

$\sim (-)^r\{ f(p - gf)\hbar - (q - fg)f\hbar \}$

$= (-)^r\{ fp\hbar - fgf\hbar - qf\hbar + fgf\hbar \}$

$\sim (-)^r\{ f\hbar - f\hbar \}$

$\underset{4\varepsilon}{\sim} 0$

(2,1)成分 $= (-)^r dg + (-)^{r-1} gd$

$\sim 0.$

(2,2)成分 $= d\hbar + gf + \hbar d$

$\sim p$

$\therefore d\Gamma + \Gamma d \underset{4\varepsilon}{\sim} \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix}.$

Γは3ε morphismでできているから. 3ε chain contraction cf. with 1.4.

$\mathcal{C}(f)$ について. $f: C \to D$

$$\begin{pmatrix} d & (-)^{r-1}f \\ 0 & d \end{pmatrix}\begin{pmatrix} d & (-)^{r}f \\ 0 & d \end{pmatrix}$$

$$= \begin{pmatrix} d^2 & (-)^r df + (-)^{r-1} fd \\ 0 & d^2 \end{pmatrix}$$

もし. $d^2 \sim_{2\varepsilon} 0$, $df \sim_{2\varepsilon} fd$ ならば $(\quad)(\quad) \sim_{2\varepsilon} 0$

例えば: C, D 共に ε chain cx.

f が ε chain map ならば

$\mathcal{C}(f)$ も ε chain complex.

Proposition 1.5 Direct sum induces an abelian group structure on $\tilde{K}_0(X, p_X, n, \varepsilon)$. Further, if $[C, p] = [C', p'] \in \tilde{K}_0(X, p_X, n, \varepsilon)$, then there is a 3ε chain equivalence

$$(C, p) \oplus (E, 1) \to (C', p') \oplus (E', 1),$$

for some n-dimensional free ε chain complexes $(E, 1)$, $(E', 1)$ on p_X. In particular, they are 3ε, n stably chain equivalent.

[証明] (A, p) : ε projective module.

$\Rightarrow (A, 1-p)$ も 〃 〃 .

$$(1-p)(1-p) = \overset{\varepsilon}{1} - \overset{\varepsilon}{p} - \overset{\varepsilon}{p} + \overset{\varepsilon}{p^2} \sim_{\varepsilon} 1 - p - p + p \quad ✓$$

$$\sim_{\varepsilon} 1 - p$$

（仮に $p^2 \sim_{2\varepsilon} p$ と def を変更しても

$(1-p)^2 \sim_{2\varepsilon} (1-p)$ ✓.）

$$\begin{pmatrix} p & 1-p \end{pmatrix} \begin{pmatrix} p \\ 1-p \end{pmatrix} = p^2 + (1-p)^2 \sim_{\varepsilon} p + 1 - p \sim_{\varepsilon} 1 \quad /$$

$$\begin{pmatrix} p \\ 1-p \end{pmatrix} \begin{pmatrix} p & 1-p \end{pmatrix} = \begin{pmatrix} p^2 & p-p^2 \\ p-p^2 & (1-p)^2 \end{pmatrix} \sim_{\varepsilon} \begin{pmatrix} p & \cancel{p-p} \\ \cancel{p-p} & 1-p \end{pmatrix} \quad ✓$$

これも def を、$p^2 \sim_{2\varepsilon} p$ にしても、OK.

$$(A, 1) \cong (A, p) \oplus (A, 1-p)$$

↑

ε isomorphism.

※忘れてはいけないのは 共に ε morphism であること!!

$$1 \begin{pmatrix} p & 1-p \end{pmatrix} = \begin{pmatrix} p & 1-p \end{pmatrix}, \quad \begin{pmatrix} p & 1-p \end{pmatrix} \begin{pmatrix} p & \\ & 1-p \end{pmatrix} \sim_{\varepsilon} \begin{pmatrix} p & 1-p \end{pmatrix} \quad ✓$$

他も同様

1.5 つづき.　$[C,p] \in \tilde{K}_0(X, p_X, n, \varepsilon)$

$$\begin{tikzcd}[column sep=large]
{} \arrow[d, "d"] & & {} \arrow[d, "0"] & {} \arrow[d, "d"] \\
(C_n, p_n) \arrow[d, "d"] & \oplus & (C_n, 1-p_n) \arrow[d, "0"] \arrow[r, shift left, "{(p_n \ \ 1-p_n)}"] & (C_n, 1) \arrow[d, "d"] \arrow[l, shift left, "{\begin{pmatrix} p_n \\ 1-p_n \end{pmatrix}}"] \\
(C_{n-1}, p_{n-1}) \arrow[d, "d"] & \oplus & (C_{n-1}, 1-p_{n-1}) \arrow[d, "0"] \arrow[r, shift left, "{(p_{n-1} \ \ 1-p_{n-1})}"] & (C_{n-1}, 1) \arrow[d, "d"] \arrow[l, shift left, "{\begin{pmatrix} p_{n-1} \\ 1-p_{n-1} \end{pmatrix}}"] \\
\vdots & & \vdots & {}
\end{tikzcd}$$

ε chain map であること.　　　　d は ε morphism だから.

⤵ $= d(p_n \ \ 1-p_n) = (dp_n \ \ d - dp_n) \sim_\varepsilon (d \ \ d-d) \sim_\varepsilon (d \ \ 0)$

↳ $= (p \ \ 1-p)\begin{pmatrix} d & \\ & 0 \end{pmatrix} = (pd \ \ 0) \sim_\varepsilon (d \ \ 0)$

∴ ⤵ $\sim_\varepsilon$ ↳

⬐ $= \begin{pmatrix} d & \\ & 0 \end{pmatrix}\begin{pmatrix} p \\ 1-p \end{pmatrix} = \begin{pmatrix} dp \\ 0 \end{pmatrix} \sim_\varepsilon \begin{pmatrix} d \\ 0 \end{pmatrix}$,　↵ $= \begin{pmatrix} p \\ 1-p \end{pmatrix} d = \begin{pmatrix} pd \\ d-pd \end{pmatrix} \sim_\varepsilon \begin{pmatrix} d \\ d-d \end{pmatrix} \sim_\varepsilon \begin{pmatrix} d \\ 0 \end{pmatrix}$

∴ ⬐ $\sim_\varepsilon$ ↵

よって →, ← 共に. ε chain map.

前頁でみたように

↩ $\sim_\varepsilon \begin{pmatrix} p & \\ & 1-p \end{pmatrix}$　　↪ $\sim_\varepsilon 1$.

∴ ε isomorphism.

$$\therefore [C,p] + [C,1-p] = \underset{\text{free}}{[C,1]} = 0 \in \tilde{K}_0(X, p_X, n, \varepsilon)$$

$[C,1-p]$ ⇠ 逆元の存在がいえた。

$[C, p] = [C', p'] \in \tilde{K}_0(X, p_X, n, \varepsilon)$ とする。

$\exists$ seq. $(C, p) = (C^{(1)}, p^{(1)}), \cdots, (C^{(m)}, p^{(m)}) = (C', p')$

$\exists$ free ε chain cx's $(E^{(k)}, 1), (F^{(k)}, 1)$

s.t.

$$(C^{(k)}, p^{(k)}) \oplus (E^{(k)}, 1) \underset{\varepsilon}{\simeq} (C^{(k+1)}, p^{(k+1)}) \oplus (F^{(k)}, 1)$$

$$(C^{(1)}, p^{(1)}) \oplus \sum_{k=1}^{n-1} (E^{(k)}, 1) \oplus \sum_{k=1}^{n} (C^{(k)}, 1)$$

$\cong$ . .

あとは論文のとおり.

1.6.

(C, p) : n-dim. ε proj. chain complex に対し. (D, q) を.

$$(D_r, q_r) = \begin{cases} \bigoplus_{i \geq r} (C_i, 1) & \text{if } r > 0 \\ \left(\bigoplus_{r:\text{even}} (C_r, p_r)\right) \oplus \left(\bigoplus_{\text{odd } r} (C_r, 1-p_r)\right) & \text{if } r = 0. \end{cases}$$

と定める。 但し. d_D は.

$$(d_D)_r = \begin{array}{c|cccc} & C_r & C_{r+1} & C_{r+2} & \cdots \\ \hline C_{r-1} & d & 0 & 0 & \cdots \\ C_r & 1-p_r & 0 & 0 & \cdots \\ C_{r+1} & 0 & p_{r+1} & 0 & \cdots \\ C_{r+2} & \vdots & & 1-p_{r+2} & \\ \vdots & & & & p_{r+3} \\ & & & & & \ddots \end{array}$$

$$: (C_r, 1) \oplus (C_{r+1}, 1) \oplus \cdots \oplus (C_n, 1) \longrightarrow \begin{cases} (C_{r-1}, 1) \oplus \cdots \oplus (C_n, 1) & r > 1 \\ (C_0, p_0) \oplus (C_1, 1-p_1) \oplus \cdots & r = 0 \end{cases}$$

主張 1 : $(D, q) = \{(D_r, q_r), d_D\}$ は. ε proj. chain cx.

① 各 (D_r, q_r) は明らかに. ε proj. modules. ✓

② d_D が ε morphism であること. ✓

③ $d_D^2 \underset{2\varepsilon}{\sim} 0$ であること.

$$(d_D)_r (d_D)_{r+1} = \begin{pmatrix} d & & & \\ 1-p_r & & & \\ & p_{r+1} & & \\ & & 1-p_{r+2} & \\ & & & \ddots \end{pmatrix} \begin{pmatrix} d & & & \\ 1-p_{r+1} & & & \\ & p_{r+2} & & \\ & & 1-p_{r+3} & \\ & & & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} d^2 & 0 & 0 & \cdots \\ (1-p_r)d & 0 & 0 & \\ p_{r+1}(1-p_{r+1}) & 0 & 0 & \\ 0 & (1-p_{r+2})p_{r+2} & 0 & \\ \vdots & 0 & p_{r+3}(1-p_{r+3}) & \\ & \vdots & 0 & \\ & & \vdots & \end{pmatrix}$$

$$= \begin{pmatrix} \boxed{d^2} \sim_{2\varepsilon} d. & 0 & 0 & \cdots \\ d - \boxed{p_r d} \sim_{\varepsilon} d. & 0 & 0 & \\ p_{r+1} - p_{r+1}^2 & 0 & 0 & \\ 0 & p_{r+2} - p_{r+2}^2 & 0 & \\ \vdots & 0 & p_{r+3} - p_{r+3}^2 & \ddots \\ & \vdots & 0 & \\ & & \vdots & \end{pmatrix}$$

$$\underset{2\varepsilon}{\sim} \begin{pmatrix} 0 & & \\ d - d & & \\ p_{r+1} - p_{r+1} & & \\ 0 & p_{r+2} - p_{r+2} & \\ \vdots & & p_{r+3} - p_{r+3} \end{pmatrix} \sim_{\varepsilon} 0$$

主張2 $\tilde{p}_r = \underset{\longrightarrow}{\begin{pmatrix} p_r \\ 0 \\ \vdots \\ 0 \end{pmatrix}}$, $\hat{p}_r = \overset{\longleftarrow}{(p_r \ 0 \ \cdots \ 0)}$ は ε chain map.

$$\begin{tikzcd}
{} \arrow[d, dashed] & {} \arrow[d, dashed, "d_D"] \\
(C_r, p_r) \arrow[r, shift left] \arrow[d, "d"] & (C_r, 1) \oplus (C_{r+1}, 1) \oplus \cdots \oplus (C_n, 1) \arrow[l, shift left] \arrow[d, "d_D"] \\
(C_{r-1}, p_{r-1}) \arrow[r, shift left] \arrow[d] & (C_{r-1}, 1) \oplus (C_r, 1) \oplus \cdots \oplus (C_n, 1) \arrow[l, shift left] \arrow[d, "d_D"] \\
\vdots \arrow[d] & \vdots \arrow[d, "d_D"] \\
(C_1, p_1) \arrow[r, shift left] \arrow[d] & (C_1, 1) \oplus (C_2, 1) \oplus \cdots \oplus (C_n, 1) \arrow[l, shift left] \arrow[d, "d_D"] \\
(C_0, p_0) \arrow[r, shift left] \arrow[d] & (C_0, p_0) \oplus (C_1, 1-p_1) \oplus \cdots \arrow[l, shift left] \arrow[d] \\
0 & 0
\end{tikzcd}$$

* 共に ε morphism は OK.

$$\text{↴} = d_D \tilde{p}_r = \begin{pmatrix} d & & & \\ 1-p_r & & & \\ & p_{r+1} & & \\ & & 1-p_{r+2} & \\ & & & \ddots \end{pmatrix} \begin{pmatrix} p_r \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} d p_r \\ (1-p_r) p_r \\ 0 \\ \vdots \end{pmatrix} \sim_{\varepsilon} \begin{pmatrix} d \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

$$\text{↳} = \begin{pmatrix} p_r \\ 0 \\ \vdots \\ 0 \end{pmatrix} d \sim_{\varepsilon} \begin{pmatrix} d \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \therefore \ \text{↴} \sim_{\varepsilon} \text{↳}$$

1.6 つづき

⬐ $= d\,(p_r\ 0\ \cdots\ 0) = (dp_r\ 0\ \cdots\ 0) \sim_\varepsilon (d\ 0\ \cdots\ 0)$

↲ $= (p_{r-1}\ 0\ \cdots\ 0)\begin{pmatrix} d & & \\ 1-p_r & & \\ & p_{r+1} & \\ & & \ddots \end{pmatrix} = (p_{r-1}d\ \ 0\ \cdots\ 0)$

$\sim_\varepsilon (d\ 0\ \cdots\ 0)$

∴ ⬐ $\sim_\varepsilon$ ↲.

よって共に εchain map.

さて. ↻ $= (p_r\ 0\ \cdots\ 0)\begin{pmatrix} p_r \\ 0 \\ \vdots \\ 0 \end{pmatrix} = p_r^2 \sim_\varepsilon p_r$

∴ $d\underset{\uparrow}{\bigcirc} + \underset{\uparrow}{\bigcirc}\, d \sim_\varepsilon P -$ ↻.

ホモトピー.

↺ $= \begin{pmatrix} p_r \\ 0 \\ \vdots \\ 0 \end{pmatrix}(p_r\ 0\ \cdots\ 0) \sim_\varepsilon \begin{pmatrix} p_r & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \text{\huge 0} & \\ 0 & & & \end{pmatrix}$

$d_D? + ?d_D \sim_\varepsilon \boxed{q} -$ ↺

$$\begin{tikzcd}
 & & (C_r,1) \arrow[dl, "d"'] \arrow[d, "1-p_r"] & \oplus (C_{r+1},1) \arrow[d, "p_{r+1}"] & \oplus \cdots \oplus & (C_n,1) \arrow[d] \\
 & (C_{r-1},1) \arrow[dl, "d"'] \arrow[d, "1-p_{r-1}"] & \oplus (C_r,1) \arrow[u, bend right, "1"'] \arrow[d, "p_r"] & \oplus (C_{r+1},1) \arrow[u, bend right, "1"'] \arrow[d, "1-p_{r+1}"] & \oplus \cdots & \oplus (C_n,1) \arrow[u, bend right, "1"'] \arrow[d] \\
(C_{r-2},1) & \oplus (C_{r-1},1) \arrow[u, bend right, "1"'] & \oplus (C_r,1) \arrow[u, bend right, "1"'] \arrow[ul, bend left, "1-p_1"] & \oplus (C_{r+1},1) \arrow[u, bend right, "1"'] \arrow[ul, bend left, "p_2"] & \oplus \cdots & \oplus (C_n,1) \arrow[u, bend right, "1"']
\end{tikzcd}$$

(又は $(C_0, p_0) \oplus (C_1, 1-p_1) \oplus (C_2, p_2) \oplus \cdots \quad \oplus \cdots$ if $r=2$.)

$? = (0\quad 1) : (D_r, q_r) = (C_r, 1) \oplus (D_{r+1}, q_{r+1}) \longrightarrow (D_{r+1}, q_{r+1})$ if $r \geqslant 1$.

$? = 0 \oplus (1-p_1) \oplus p_2 \oplus \cdots : (C_0, p_0) \oplus (C_1, 1-p_1) \oplus \cdots \oplus \quad \to 0 \oplus (C_1, 1) \oplus \cdots \oplus (C_n, 1)$

$$d? + ?d = \begin{pmatrix} 1-p_r & & & \\ & p_{r+1} + (1-p_{r+1}) & & \\ & & (1-p_{r+2}) + p_{r+2} & \\ & & & \ddots \end{pmatrix}$$

$$: (C_r, 1) \oplus (C_{r+1}, 1) \oplus \cdots \oplus (C_n, 1) \circlearrowleft$$

$$\sim_{\varepsilon} \begin{pmatrix} 1-p_r & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} - \begin{pmatrix} p_r & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \bigcirc & \\ 0 & & & \end{pmatrix} = g - G .$$

一番下では

$$d? + ?d = \begin{pmatrix} 0 & & & & \\ & (1-p_1)^2 & & & \\ & & p_2^2 & & \\ & & & (1-p_3)^2 & \\ & & & & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} 0 & & & & \\ & 1-p_1 & & & \\ & & p_2 & & \\ & & & 1-p_3 & \\ & & & & \ddots \end{pmatrix}$$

$$\sim_{\varepsilon} \begin{pmatrix} p_0 & & & & \\ & 1-p_1 & & & \\ & & p_2 & & \\ & & & 1-p_3 & \\ & & & & \ddots \end{pmatrix} - \begin{pmatrix} p_0 & & \\ & & \\ & & \end{pmatrix} = g_0 - G .$$

Corollary 1.7. $\tilde{K}_0(X, p_X, \varepsilon) \longrightarrow \tilde{K}_0(X, p_X, n, \varepsilon)$ is onto.

(C, p)に対し. 1.6の (D, q)をとる.

$$\begin{tikzcd}
{} \arrow[d, dashed] & & & \vdots \arrow[d] \\
(C_2, p_2) \arrow[d, "d"'] \arrow[rrr, "\tilde{p}_2 ="] & & & (D_2, 1) = (C_2,1)\oplus(C_3,1)\oplus\cdots\oplus(C_n,1) \arrow[d, "d_D"] \\
(C_1, p_1) \arrow[d, "d"'] \arrow[rrr, "\tilde{p}_1 ="] \arrow[dr, dashed, "0"] & & & (D_1, 1) = (C_1,1)\oplus(C_2,1)\oplus\cdots\oplus(C_n,1) \arrow[d, "d_D"] \arrow[dl, dashed, "0"'] \\
(C_0, p_0) \arrow[d] & \oplus (D_0, 1) \arrow[r, "\begin{pmatrix} 0 & q_0 \\ \tilde{p}_0 & 1-q_0 \end{pmatrix}"'] & (D_0, q_0) & \oplus (D_0, 1) \arrow[d] \\
0 & & & 0
\end{tikzcd}$$

これが ε chain map であること.

$r>1$ では. ⬎ $= d_D \tilde{p}_r$, ↳ $= \tilde{p}_{r-1} d$

1.6により. ⬎ $\sim_\varepsilon$ ↳ .

$r=1$ では. ⬎ $= \begin{pmatrix} 0 \\ d_D \end{pmatrix} \tilde{p}_1 = \begin{pmatrix} 0 \\ d_D \tilde{p}_1 \end{pmatrix}$, ↳ $= \begin{pmatrix} 0 & q_0 \\ \tilde{p}_0 & 1-q_0 \end{pmatrix} \begin{pmatrix} d \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{p}_0 d \end{pmatrix}$

∴ ⬎ $\sim_\varepsilon$ ↳ ✓

$r=0$ は OK.

よって. ε chain map である。

逆方向.

$$\begin{tikzcd}
{} \arrow[d] & & \vdots \arrow[d] \\
{} & & {} \arrow[ll, "\hat{p}_2"'] \\
(C_1, p_1) \arrow[d] & & (D_1, 1) \arrow[ll, "\hat{p}_1"'] \arrow[d] \\
(C_0, p_0) \oplus (D_0, 1) \arrow[d] & & (D_0, q_0) \oplus (D_0, 1) \arrow[ll, "\begin{pmatrix} 0 & \hat{p}_0 \\ q_0 & 1-q_0 \end{pmatrix}"] \\
0 & &
\end{tikzcd}$$

Reviewerの意見

d_D を

$(D_1, 1) \longrightarrow (D_0, 1)$) はよくない

$$\begin{tikzcd}
(D_1, 1) \arrow[r, "d_D"] \arrow[rr, bend right, "g_0 d_D"'] & (D_0, g_0) \arrow[r, "g_0"] & (D_0, 1)
\end{tikzcd}$$

としたほうがよい.

但し. $g_0 d_D \sim d_D$

これが chain map であること

$r>1$ では 1.6 より明らか.

$r=1$ では.

$$↴ = \begin{pmatrix} d \\ 0 \end{pmatrix}\hat{P}_1 = \begin{pmatrix} d\hat{P}_1 \\ 0 \end{pmatrix} \sim_\varepsilon \begin{pmatrix} \hat{P}_0 d_D \\ 0 \end{pmatrix}$$

$$↵ = \begin{pmatrix} 0 & \hat{P}_0 \\ g_0 & 1-g_0 \end{pmatrix}\begin{pmatrix} 0 \\ d_D \end{pmatrix} = \begin{pmatrix} \hat{P}_0 d_D \\ (1-g_0)d_D \end{pmatrix} \underset{\uparrow}{\sim_\varepsilon} \begin{pmatrix} \hat{P}_0 d_D \\ 0 \end{pmatrix}$$

<u>d_D は ε morphism だから</u>

合成は?

$$r=0 \quad ↻ = \begin{pmatrix} 0 & \hat{P}_0 \\ g_0 & 1-g_0 \end{pmatrix}\begin{pmatrix} 0 & g_0 \\ \tilde{P}_0 & 1-g_0 \end{pmatrix} = \begin{pmatrix} \hat{P}_0\tilde{P}_0 & \hat{P}_0(1-g_0) \\ (1-g_0)\tilde{P}_0 & g_0^2+(1-g_0)^2 \end{pmatrix}$$

$$\sim_\varepsilon \begin{pmatrix} P_0 & 0 \\ 0 & g_0+(1-g_0) \end{pmatrix} \sim_\varepsilon \begin{pmatrix} P_0 & 0 \\ 0 & 1 \end{pmatrix} = \textcircled{$P_0\oplus 1$} \ . \ \checkmark$$

ホモトピーは0でよい.

$r>0$ では $\textcircled{$P_r$}$

<u>$r=0$</u>

$$↺ = \begin{pmatrix} 0 & g_0 \\ \tilde{P}_0 & 1-g_0 \end{pmatrix}\begin{pmatrix} 0 & \hat{P}_0 \\ g_0 & 1-g_0 \end{pmatrix} = \begin{pmatrix} g_0^2 & g_0(1-g_0) \\ (1-g_0)g_0 & \tilde{P}_0\hat{P}_0+(1-g_0)^2 \end{pmatrix}$$

$$\sim_\varepsilon \begin{pmatrix} g_0 & 0 \\ 0 & \boxed{*} \end{pmatrix}$$

但し
$$\boxed{*} = \begin{vmatrix} P_0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{vmatrix} + 1 - g_0 = 1 + \begin{vmatrix} P_0 & & \\ & & \\ & & \end{vmatrix} - \begin{pmatrix} P_0 & & & \\ & 1-P_1 & & \\ & & P_2 & \\ & & & \ddots \end{pmatrix}$$

$$\sim_\varepsilon 1 - \begin{pmatrix} 0 & & & & \\ & 1-P_1 & & & \\ & & P_2 & & \\ & & & 1-P_3 & \\ & & & & \ddots \end{pmatrix} \left(\text{又は} \begin{pmatrix} 1 & & & & \\ & P_1 & & & \\ & & 1-P_2 & & \\ & & & P_3 & \\ & & & & \ddots \end{pmatrix} \right)$$

$r>0$ では.
$$\begin{pmatrix} P_r & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 0 \end{pmatrix} .$$

$$\begin{tikzcd}
 & {} \arrow[d] & \\
 & (D_2,1) \arrow[d, "d_D"'] & \\
 & (D_1,1) \arrow[d, "d_D"'] \arrow[u, dashed, bend right] & \\
(D_0,q_0) \arrow[ur, dashed, "0"] & \oplus (D_0,1) \arrow[d] \arrow[u, dashed, bend right] & \\
 & 0 &
\end{tikzcd}$$

1.6 と同じ morphisms.

すると. $d\square + \square d$ は..

$r>0$ では $\underset{\varepsilon}{\sim} 1 - \begin{pmatrix} p_r & \\ & 0 \end{pmatrix}$ ✓.

$r=0$ では. $= 0 \oplus \begin{pmatrix} 0 & & & & \\ & 1-p_1 & & & \\ & & p_2 & & \\ & & & 1-p_3 & \\ & & & & \ddots \end{pmatrix}$

$= q_0 \oplus 1 - q_0 \oplus \begin{pmatrix} 1 & & & & \\ & p_1 & & & \\ & & 1-p_2 & & \\ & & & p_3 & \\ & & & & \ddots \end{pmatrix}$

$\underset{\varepsilon}{\sim} q_0 \oplus 1 - G$. ✓

よって. ε chain equivalence.

Proposition 1.8 $(C,p) \mapsto (D_0, \delta_0)$ defines a well-defined homomorphi

$$\sigma : \tilde{K}_0(X, p_X, n.\varepsilon) \longrightarrow \tilde{K}_0(X, p_X, 9\varepsilon)$$

証明. 特別な場合 $(C,p) \simeq_\varepsilon 0$. i.e. (C,p) is ε chain contractible.

Γ : ε chain contraction とおく.

$\Gamma_r : (C_r, p_r) \longrightarrow (C_r, p_{r+1})$ ε morphism.

$d\Gamma + \Gamma d \underset{2\varepsilon}{\sim} p_r$.

$\Gamma' = \Gamma d \Gamma$ とおく. 3ε morphism.

$$d\Gamma' + \Gamma' d = d\Gamma \boxed{d\Gamma} + \boxed{\Gamma d} \Gamma d \underset{4\varepsilon}{\sim} p$$

$$\begin{aligned} (\because) \quad &\underset{4\varepsilon}{\sim} d\Gamma(p - \Gamma d) + (p - d\Gamma)\Gamma d \\ &= d\Gamma p - d\Gamma\Gamma d + p\Gamma d - d\Gamma\Gamma d \\ &\underset{4\varepsilon}{\sim} d\Gamma + \Gamma d \underset{2\varepsilon}{\sim} p . \end{aligned}$$

$\therefore$ Γ' は 3ε chain contraction

$$\begin{aligned} \Gamma'^2 &= \Gamma d \Gamma \Gamma d \Gamma \\ &\underset{6\varepsilon}{\sim} \Gamma(p - \Gamma d)\Gamma d \Gamma = \Gamma p \Gamma d \Gamma - \Gamma\Gamma d \boxed{\Gamma d} \Gamma \\ &\underset{6\varepsilon}{\sim} \Gamma\Gamma d\Gamma - \Gamma\Gamma d(p - d\Gamma)\Gamma = \Gamma\Gamma d\Gamma - \Gamma\Gamma d p \Gamma + \Gamma\Gamma d^2 \Gamma\Gamma \\ &\underset{6\varepsilon}{\sim} \Gamma\Gamma d \Gamma - \Gamma\Gamma d\Gamma \underset{4\varepsilon}{\sim} 0 \end{aligned}$$

$\therefore$ $\Gamma'^2 \underset{6\varepsilon}{\sim} 0$.

$\therefore$ $(d + \Gamma')(d + \Gamma') = d^2 + d\Gamma' + \Gamma' d + \Gamma'^2 \underset{6\varepsilon}{\sim} p$

$\therefore$ $d + \Gamma'$ は 3ε isomorphism $\therefore$ $[D_0, \delta_0] = 0$ in $\tilde{K}_0(X, p_X, 3\varepsilon)$.

一般

① $f:(C,p)\to(C',p')$ ε chain eq. とする.

⇒ $\mathcal{C}(f)$ は 3ε contractible. (by 1.4)

⇒ 前半より.

$$\sum_r (-1)^r [\mathcal{C}(f)_r, p'_r \oplus p_{r-1}] = 0 \in \tilde{K}_0(X, p_X, 9\varepsilon).$$

$$\parallel$$

$$\sum_r (-1)^r [C'_r, p'_r] - \sum_r (-1)^r [C_r, p_r]$$

∴

② ⊕(free) ✓

Corollary 1.9

$$n>0, \quad [C,p]=0 \in \widetilde{K}_0(X,p_X,n,\varepsilon) \longrightarrow \begin{array}{c} \widetilde{K}_0(X,p_X,9\varepsilon) \\ \rotatebox{90}{$\in$} \\ [D_0,q_0]=0. \\ \shortparallel \\ \left(\ \sum(-1)^r[C_r,p_r]\ \right). \end{array}$$

$$(C,p) \rightsquigarrow (D,q)$$

1.5により、

$$(D_0,q_0)\oplus(F,1) \underset{27\varepsilon}{\cong} {}^{\exists}(G,1)$$

$$(C,p) \underset{\varepsilon}{\simeq} (D,q) \overset{(1)}{\underset{\varepsilon}{\simeq}} (D,q)\oplus(\cdots\to 0\to(F,1)\xrightarrow{1}(F,1)\to 0\to\cdots)$$

$$\overset{(2)}{\underset{55\varepsilon}{\simeq}} \text{ a } 28\varepsilon \text{ free chain complex.}$$

合成で $\underset{57\varepsilon}{\simeq}$.

(1): 一般に $(C,p)\underset{\varepsilon}{\simeq}(C',p')$, $(D,q)\underset{\delta}{\simeq}(D',q')$

$$\Rightarrow (C,p)\oplus(D,q) \underset{\max\{\varepsilon,\delta\}}{\simeq} (C',p')\oplus(D',q').$$

$$0 \underset{\boxed{?}}{\simeq} \left(\begin{tikzcd} \to 0 \arrow[r] & (F,1) \arrow[r, "d=1"] & (F,1) \arrow[r] \arrow[l, bend left, "\mathcal{R}=1"] & 0 \end{tikzcd} \right)$$

$$d\mathcal{R}+\mathcal{R}d \underset{\uparrow}{=} 1.$$

本当に.

任意の正数が使える!!

$(D,q)\underset{\varepsilon}{\simeq}(D,q)$ だから. (1)はOK.

(2)

$$\begin{tikzcd}
\vdots \arrow[d, "d_D"] & & \vdots \arrow[d, "d_D"] & \\
(D_2,1) \arrow[d, "d_D"] \arrow[dr, dashed, "0"] \arrow[rr, equal] & & (D_2,1) \arrow[d, "d_D"] \arrow[dr, dashed, "0"] & \\
(D_1,1) \arrow[d, "d_D"] & \oplus(F,1) \arrow[d, "1"] \arrow[r, equal] & (D_1,1)\oplus(F,1) \arrow[d, dashed, "d'"] & \\
(D_0,q_0) \arrow[d] & \oplus(F,1) \arrow[r, shift left, "f"] & (G,1) \arrow[l, shift left, "f^{-1}"] \arrow[d] & \\
0 & 27\varepsilon \text{ iso.} & 0. &
\end{tikzcd}$$

$$d' = \underset{27\varepsilon\text{ morph}}{\boxed{f}}\,\big(\underset{\varepsilon\text{ morph.}}{d_D \oplus 1}\big) \quad : \ 28\varepsilon \text{ morphism.}$$

$$\underbrace{f}_{27\varepsilon.}\begin{pmatrix} d_D & \\ & 1\end{pmatrix}\begin{pmatrix} d_D \\ 0\end{pmatrix} = f\begin{pmatrix} \overset{\sim_{2\varepsilon}0}{d_D^2} \\ 0\end{pmatrix} \sim_{29\varepsilon} 0 .$$

∴ 右は 28ε free chain complex.

→ ⊛ ⤵ = ↳

正確には

⤵ $\sim_{28\varepsilon}$ ↳ .

∴ 28ε chain map

← ∘ ⬐ $= \begin{pmatrix} d_D & \\ & 1 \end{pmatrix}$

$$↲ = f^{-1}d' = \overset{\varepsilon}{f^{-1}f\begin{pmatrix} d_D & \\ & 1\end{pmatrix}} \sim_{55\varepsilon} \begin{pmatrix} d_D & \\ & 1\end{pmatrix}$$

∴ ← は 55ε chain map.

⟲ = 1 or $f^{-1}f \sim_{54\varepsilon} 1$.

⟳ = 1 or $f f^{-1} \sim_{54\varepsilon} 1$.

∴ 55ε isomorphism

§4.

Notation, Terminology

- ε deformation の合成を ε-simple isomorphism という.

- $f: C \longrightarrow D$, $f^{-1}: D \longrightarrow C$ $f|C_i : C_i \to D_i$, $f^{-1}|D_i : D_i \to C_i$
  共に { ε-simple isomorphism, ε chain map.

  のとき f は ε-simple isomorphism

正確には: 1. f: ε chain map.
2. 各 $f_r : C_r \longrightarrow D_r$ は ε-simple isomorphism.
（従って $f_r^{-1} : D_r \longrightarrow C_r$ が自然に定まる.）
3. $f^{-1} : D \longrightarrow C$ が ε chain map

$f : C \underset{\varepsilon,\Sigma}{\cong} D$ 必要になったら導入する.

$C \overset{\exists}{\oplus} T \underset{\varepsilon,\Sigma}{\cong} C' \overset{\exists}{\oplus} T'$ のとき $C \overset{n}{\underset{\varepsilon}{\sim}} C'$
n-stably ε equivalent.

ε を忘れれば同値関係.

$C \overset{n,W}{\underset{\varepsilon}{\sim}} C' \iff {}^{\exists}D, D'$: n-dim. free ε chain complexes on p_{X-W}
s.t. $C \oplus D \overset{n}{\underset{\varepsilon}{\sim}} C' \oplus D'$.

(n, W)-stably ε equivalent

$\therefore$ (n, X)-stable ε equivalence = n-stable ε equivalence.

$C \overset{n,W}{\underset{\varepsilon}{\sim}} C'$, $C' \overset{m,V}{\underset{\delta}{\sim}} C'' \Rightarrow C \overset{\max\{n,m\},\, W \cap V}{\underset{\cancel{\max\{\varepsilon,\delta\}}}{\sim\!\sim\!\sim}} C''$.

? $\varepsilon + \delta$?

Γ : a δ chain contraction of C over W

Γ : δ morphism, $d\Gamma$, Γd : 2δ morphism 追加.

$(d\Gamma + \Gamma d)|W \underset{2\delta}{\sim} 1$

$\Gamma' = \Gamma d \Gamma$ とおく : 3δ morphism.

(∵ $d\Gamma$ に関する上の追加仮定より)

$$\Gamma'^2 = \Gamma d \Gamma \Gamma d \Gamma$$

$$\begin{aligned}
\Gamma'^2 | W^{-3\delta} &= (\Gamma d \Gamma \Gamma d \Gamma)|W^{-3\delta} \\
&\underset{6\delta}{\sim} \Gamma(1-\Gamma d)\Gamma d \Gamma \,|\, W^{-3\delta} \\
&= \Gamma\Gamma d \Gamma | W^{-3\delta} - \Gamma\Gamma d \boxed{\Gamma} d \Gamma \,|W^{-3\delta} \\
&\underset{6\delta}{\sim} \Gamma\Gamma d\Gamma | W^{-3\delta} - \Gamma\Gamma d(1-d\Gamma)\Gamma \,|W^{-3\delta} \\
&= \Gamma\Gamma d\Gamma | W^{-3\delta} - \Gamma\Gamma d\Gamma | W^{-3\delta} + \Gamma\Gamma \underbrace{dd}\Gamma\Gamma | W^{-3\delta} \\
&\underset{6\delta}{\sim} 0
\end{aligned}$$

Prop. 2.1

$Wh(X, Y, p_X, n, \varepsilon) \ni [C]$ 逆元は？

Lemma 2.2 $C = \{C_r, d_r\}$ a free ε ch. cx of dim n on p_X

$C' = \{C'_r, d'_r\}$ ε' n'

$W \supset X$

1. C has an ε chain contraction Γ over $X - W^{\varepsilon}$
2. $C_r(X-W) = C'_r(X-W)$ $\forall r$.
3. $d_r | X - W^{\varepsilon} = d'_r | X - W^{\varepsilon} : C_r(X - W^{\varepsilon}) \to C_{r-1}(X-W)$ $\forall r$

$\delta = \min\{\varepsilon, \varepsilon'\}$, $\gamma = \max\{\varepsilon, \varepsilon'\}$

$m = \max\{n+1, n'\}$.

$\Rightarrow$ $(3\varepsilon + 3\delta + \gamma)$-simple isom : $C' \oplus \Sigma C \overset{\simeq}{\Rightarrow}$ (an m-dim free $2\varepsilon + 2\delta + \gamma$ ch. cx on $p_{W^{7\varepsilon+4\delta+2\gamma}}$)

$\oplus$

(an m-dim trivial ch. cx on p_X).

証明.

$\hat{d}_r : C'_r \longrightarrow C_{r-1}$ (radius δ を持つ)

$$\hat{d}_r = \begin{cases} d_r \overset{\text{ここでは}}{=} d'_r & \text{over } X - W^{\varepsilon} \\ 0 & \text{over } W^{\varepsilon}. \end{cases}$$

$\hat{\Gamma}_r : C_r \longrightarrow C'_{r+1}$ (radius ε をもつ)

$$\hat{\Gamma}_r = \begin{cases} \Gamma_r & \text{over } X - W^{\varepsilon} \\ 0 & \text{over } W^{\varepsilon}. \end{cases}$$

とおく。

$$f_r = \begin{pmatrix} (-)^r 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (-)^{r-1}\hat{d} & 1 \end{pmatrix} \begin{pmatrix} 1 & (-)^r\hat{\Gamma} \\ 0 & 1 \end{pmatrix} = \underset{\substack{\uparrow \\ \text{最初の2つの積}}}{\begin{pmatrix} (-)^r 1 & 0 \\ (-)^{r-1}\hat{d} & 1 \end{pmatrix}} \begin{pmatrix} 1 & (-)^r\hat{\Gamma} \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (-)^r 1 & \hat{\Gamma} \\ (-)^{r-1}\hat{d} & -\hat{d}\hat{\Gamma} + 1 \end{pmatrix}$$

$$\text{後の2つの積} = \begin{pmatrix} 1 & (-)^r\hat{\Gamma} \\ (-)^{r-1}\hat{d} & -\hat{d}\hat{\Gamma} + 1 \end{pmatrix}$$

又. それらの$(\ )^{-1}$のどの積も 2ε morphism.（第1項は. sizeに影響をおよぼしていない.）

$\therefore$ f_r は $(\varepsilon+\delta)$-simple isomorphism

$$f_r^{-1} = \begin{pmatrix} 1 & (-)^{r+1}\hat{\Gamma} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (-)^r \hat{d} & 1 \end{pmatrix} \begin{pmatrix} (-)^r 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} (-)^r (1-\hat{\Gamma}\hat{d}) & (-)^{r+1}\hat{\Gamma} \\ \hat{d} & 1 \end{pmatrix}$$

$$\begin{tikzcd}
C' \oplus \Sigma C \arrow[d, dashed] & & \bar{C} \arrow[d] \\
C'_{r+1} \oplus C_r \arrow[d, "d'_{r+1} \oplus d_r"] \arrow[rr, shift left, "f_{r+1}"] & & C'_{r+1} \oplus C_r \arrow[d, "\bar{d}_{r+1}"] \arrow[ll, shift left, "f_{r+1}^{-1}"] \\
C'_r \oplus C_{r-1} \arrow[d, "d'_r \oplus d_{r-1}"] \arrow[rr, shift left, "f_r"] & & C'_r \oplus C_{r-1} \arrow[d, "\bar{d}_r"] \arrow[ll, shift left, "f_r^{-1}"] \\
C'_{r-1} \oplus C_{r-2} \arrow[d, "d'_{r-1} \oplus d_{r-2}"] \arrow[rr, shift left, "f_{r-1}"] & & C'_{r-1} \oplus C_{r-2} \arrow[d] \arrow[ll, shift left, "f_{r-1}^{-1}"] \\
\vdots & & \vdots
\end{tikzcd}$$

可換となるように. $\bar{d}$ を定める。

$\bar{d}_r = f_{r-1}(d'_r \oplus d_{r-1}) f_r^{-1}$ ： 単純には $(\varepsilon+\delta) + \gamma + (\varepsilon+\delta)$

$= 2\varepsilon + 2\delta + \gamma$.

$$= \begin{pmatrix} (-)^{r-1} 1 & \hat{\Gamma} \\ (-)^r \hat{d} & -\hat{d}\hat{\Gamma}+1 \end{pmatrix} \begin{pmatrix} d' & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} (-)^r (1-\hat{\Gamma}\hat{d}) & (-)^{r+1}\hat{\Gamma} \\ \hat{d} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (-)^{r-1} d' & \hat{\Gamma} d \\ (-)^r \hat{d} d' & -\hat{d}\hat{\Gamma} d + d \end{pmatrix} \begin{pmatrix} (-)^r (1-\hat{\Gamma}\hat{d}) & (-)^{r+1}\hat{\Gamma} \\ \hat{d} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -d'(1-\hat{\Gamma}\hat{d}) + \hat{\Gamma} d \hat{d} & d'\hat{\Gamma} + \hat{\Gamma} d \\ \hat{d} d'(1-\hat{\Gamma}\hat{d}) - \hat{d}\hat{\Gamma} d \hat{d} + d\hat{d} & -\hat{d} d' \hat{\Gamma} - \hat{d}\hat{\Gamma} d + d \end{pmatrix}$$

$$= \begin{pmatrix} -d' + (\overset{\varepsilon'}{d'}\overset{\delta}{\hat{\Gamma}} + \overset{\varepsilon}{\hat{\Gamma}}\overset{\varepsilon}{d})\overset{\delta}{\hat{d}} & d'\hat{\Gamma} + \hat{\Gamma} d \\ \hat{d} d' + d\hat{d} - \underset{\delta}{\underset{\uparrow}{\hat{d}}}(\underset{\varepsilon'+\varepsilon}{d'\hat{\Gamma}} + \underset{\varepsilon+\varepsilon}{\hat{\Gamma} d})\underset{\delta}{\underset{\uparrow}{\hat{d}}} & d - \underset{\delta}{\hat{d}}(\underset{\gamma}{d'}\underset{\varepsilon}{\hat{\Gamma}} + \underset{\varepsilon}{\hat{\Gamma}}\underset{\gamma}{d}) \end{pmatrix}$$

$(\varepsilon + 2\delta + \gamma)$- morphism である.

実際には $\underline{2\varepsilon+\delta}$ まで下げられる。（不使用）

$$\bar{d}_r \bar{d}_{r+1}$$

$$= \underbrace{f_{r-1}}_{\varepsilon+\delta} (\underbrace{d'_r \oplus d_{r-1}}_{\gamma}) \underbrace{f_r^{-1} f_r}_{\substack{\wr_{2(\varepsilon+\delta)} \\ 1}} (\underbrace{d'_{r+1} \oplus d_r}_{\gamma}) \underbrace{f_{r+1}^{-1}}_{\varepsilon+\delta}$$

$$\underset{4(\varepsilon+\delta)+2\gamma}{\sim} f_{r-1} (d'_r \oplus d_{r-1})(d'_{r+1} \oplus d_r) f_{r+1}^{-1}$$

$$= \underbrace{f_{r-1}}_{\varepsilon+\delta} (\underbrace{d'_r d'_{r+1} \oplus d_{r-1} d_r}_{\wr_{2\gamma}}) \underbrace{f_{r+1}^{-1}}_{\varepsilon+\delta}.$$

$$\underset{2(\varepsilon+\delta+\gamma)}{\sim} \quad 0$$

$$\therefore \bar{d}_r \bar{d}_{r+1} \underset{4(\varepsilon+\delta)+2\gamma}{\sim} 0.$$

$$↴ = \bar{d}_r f_r = \overset{\varepsilon+\delta}{f_{r-1}} (\overset{\gamma}{\overbrace{d'_r \oplus d_{r-1}}}) \overset{\substack{1 \\ \wr_{2(\varepsilon+\delta)}}}{\overbrace{f_r^{-1} f_r}} \underset{3(\varepsilon+\delta)+\gamma}{\sim} f_{r-1} (d'_r \oplus d_{r-1})$$

$$↳ = f_{r-1} (d'_r \oplus d_{r-1})$$

$$\therefore ↴ \underset{3(\varepsilon+\delta)+\gamma}{\sim} ↳$$

実はこの $\bar{d}$ はつかめない。

$\underline{\varepsilon}$ chain contraction

$\underline{X-W^{2\varepsilon}}$ 上で $d'\hat{\Gamma} + \hat{\Gamma} d = d\Gamma + \Gamma d \underset{2\varepsilon}{\sim} 1$

$\hat{d}$ の radius は δ だから。　　ここでは $d' = d \doteqdot \hat{d}$

$\underline{\underline{X-W^{3\varepsilon} \text{上で}}}$

$$\bar{d} = \begin{pmatrix} -\hat{d} + \overset{2\varepsilon}{(d\Gamma+\Gamma d)} \overset{\delta}{\hat{d}} & d\hat{\Gamma} + \hat{\Gamma} d \\ \underset{2\delta}{\hat{d}\hat{d}} + \underset{2\delta}{\hat{d}\hat{d}} - \underset{\delta}{\hat{d}} \underset{2\varepsilon}{(d\Gamma+\Gamma d)} \underset{\delta}{\hat{d}} & \hat{d} - \underset{\delta}{\hat{d}} \underset{2\varepsilon}{(d\Gamma+\Gamma d)} \end{pmatrix}$$

$$\underset{2\delta+2\varepsilon}{\sim} \begin{pmatrix} -\hat{d} + \hat{d} & 1 \\ \hat{d}\hat{d} + \hat{d}\hat{d} - \hat{d}\hat{d} & \hat{d} - \hat{d} \end{pmatrix} \qquad \therefore \bar{d} \underset{2\delta+2\varepsilon}{\sim} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ over } X - W^{3\varepsilon}.$$

$$\underset{2\delta}{\sim} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

そこで

$$\tilde{d} = \begin{cases} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \text{over } X - W^{3\varepsilon} \\ \\ \bar{d} & \text{over } W^{3\varepsilon} \end{cases}$$ とおく。

今の議論により、 共に $(\varepsilon + 2\delta + \gamma)$- morphism

$$\boxed{\bar{d}} \sim \boxed{\tilde{d}}$$

$\max\{\varepsilon+2\delta+\gamma,\ 2\delta+2\varepsilon\} = \varepsilon+2\delta+\gamma.$

$\varepsilon+\delta$

$$\begin{tikzcd} C' \oplus \Sigma C \arrow[r, shift left, "f"] & \tilde{C} \arrow[l, shift left, "f^{-1}"] \end{tikzcd}$$

$$\text{↴} = \overset{\varepsilon+2\delta+\gamma}{\overbrace{\tilde{d}}}\ \overset{\varepsilon+\delta}{\overbrace{f}} \underset{2\varepsilon+3\delta+\gamma}{\sim} \bar{d} f \underset{3\varepsilon+3\delta+\gamma}{\sim} f(d' \oplus d) = \text{↳}\,.$$

$$\therefore\ \text{↴} \underset{3\varepsilon+3\delta+\gamma}{\sim} \text{↳}$$

$\therefore$ f は $(3\varepsilon + 3\delta + \gamma)$ chain map

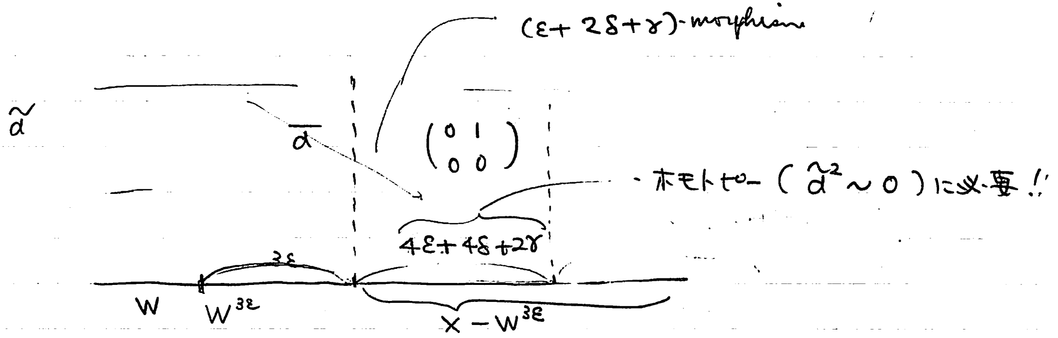
逆方向は.

$\text{⬐} = (d' \oplus d) f^{-1}$

$$\text{↲} = \underbrace{f^{-1}}_{\varepsilon+\delta} \tilde{d} \underset{2\varepsilon+3\delta+\gamma}{\sim} f^{-1} \bar{d} = \overbrace{f^{-1} f}^{\overset{1}{\wr}\,2\varepsilon+2\delta} (\underbrace{d' \oplus d}_{\gamma}) \underbrace{f^{-1}}_{\varepsilon+\delta} \underset{3\varepsilon+3\delta+\gamma}{\sim} (d' \oplus d) f^{-1}$$

よって逆 f^{-1} も $(3\varepsilon + 3\delta + \gamma)$ chain map

$\therefore$ f は $(3\varepsilon + 3\delta + \gamma)$- simple isomorphism : $C' \oplus \Sigma C \to \tilde{C}$.



$(\varepsilon+2\delta+\gamma)$-morphism

・ホモトピー ($\tilde{d}^2 \sim 0$) に必要!!

$$\tilde{d}^2 \underset{\substack{2(\varepsilon+2\delta+\gamma) \\ \| \\ 4\varepsilon+4\delta+2\gamma}}{\sim} \bar{d}^2 \underset{4\varepsilon+4\delta+2\gamma}{\sim} 0 .$$

$$\therefore \tilde{d}^2 \underset{4\varepsilon+4\delta+2\gamma}{\sim} 0$$

$\therefore$ $(\tilde{C}, \tilde{d})$ は $(2\varepsilon+2\delta+\gamma)$- chain complex.

$$\tilde{C} = \underbrace{\tilde{C}(W^{7\varepsilon+4\delta+2\gamma})}_{\text{a free } (2\varepsilon+2\delta+\gamma) \text{ chain complex}} \oplus \underbrace{\tilde{C}(X-W^{7\varepsilon+4\delta+2\gamma})}_{\text{trivial complex.}}$$

$\therefore C' \oplus \Sigma C$ is $(m, X-W^{7\varepsilon+4\delta+2\gamma})$-stably $3\varepsilon+3\delta+\gamma$ equivalent to ‥

($\tilde{C}(W^{4\varepsilon+2\delta+\gamma}) \oplus \tilde{C}(X-W^{4\varepsilon+2\delta+\gamma})$ でも定義できる。
但し. $\tilde{C}(W^{4\varepsilon+2\delta+\gamma})$ は $m\,\phi_{W^{7\varepsilon+4\delta+2\gamma}}$ (ホモトピーのため).
だから. 特に. こうする必要はない.)

$$f = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \qquad f^{-1} = \begin{pmatrix} 1 & -h \\ 0 & 1 \end{pmatrix}$$

$$f f^{-1} = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -h \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -h+h \\ 0 & 1 \end{pmatrix} \sim 1 .$$

sizeは hの sizeによる。

If h is a geometric morphism of radius ε,
(ε morphism)
then f above is an ε isomorphism.

Lemma 2.3.

C : an n-dim free ε chain complex

Γ : a strong ε chain contraction over $X-Y$. of C.

$$\begin{tikzcd} 0 \arrow[r] & C_n \arrow[r, "d"'] & C_{n-1} \arrow[l, bend right, "\Gamma"'] \arrow[r] & \cdots \arrow[r] & C_0 \arrow[r] & 0 . \end{tikzcd}$$

$$\begin{tikzcd} C_n \arrow[r, phantom, "="] \arrow[r, bend right, "r"'] \arrow[rr, bend right=60, "q"'] & C_n(Y) \arrow[l, bend right, "i"'] \arrow[r, phantom, "\oplus"] & C_n(X-Y) \arrow[ll, bend right=60, "j"'] \end{tikzcd}$$

$$\begin{tikzcd} C' : \cdots \arrow[r] & 0 \arrow[r] & C_n(Y) \arrow[r, "\Delta"] & C_{n-1} \arrow[r, "\binom{d}{q\Gamma}"] & C_{n-2} \oplus C_n(X-Y) \arrow[r, "(d\ 0)"] & C_{n-3} \arrow[r, "d"] & \cdots \arrow[r, "d"] & C_0 \arrow[r] & 0 \end{tikzcd}$$

$$\Delta = \underset{\varepsilon}{d_n} i - \underset{\varepsilon}{d_n}\underset{1}{j}\underset{0}{q}\underset{\varepsilon}{\Gamma}\underset{\varepsilon}{d_n} i \quad : \quad \underline{3\varepsilon \text{ morphism}}$$

$\left[\begin{array}{l} \text{More precisely, } i = 1_{C_n(Y)} \text{ viewed as a morphism from } C_n(Y) \text{ to } C_n \\ r = 1_{C_n(Y)} \text{ viewed as a morphism from } C_n \text{ to } C_n(Y). \end{array}\right.$

$\textcircled{d^2}$ $$\binom{d}{q\Gamma}\Delta = \begin{pmatrix} d d i - \overset{\varepsilon}{d}\overset{\varepsilon}{d} j q \overset{\varepsilon}{\Gamma}\overset{\varepsilon}{d} i \underset{4\varepsilon}{\sim} 0 \\ q\Gamma d i - \underline{q\Gamma d_n j} q\Gamma d i \end{pmatrix} \underset{4\varepsilon}{\sim} 0 .$$

$\because$

$\Gamma d_n \sim 1$ over $X-Y$ εいれかえると、

$\Gamma d j \sim j : C_n(X-Y) \longrightarrow C_n$

$\Rightarrow (q\Gamma d j \underset{2\varepsilon}{\sim} 1 : C_n(X-Y) \to C_n(X-Y)\ )$

$(d\ 0)\binom{d}{q\Gamma} = d^2 \underset{2\varepsilon}{\sim} 0$, あとは自明

$$\begin{tikzcd}
C: & 0 \arrow[r] & C_m \arrow[r, "d"] \arrow[ddd, "f_m"'] & C_{m-1} \arrow[r, "d"] & C_{m-2} \arrow[r, "d"] & C_{m-3} \arrow[r] \arrow[ddd, equal] & \cdots \\
 & & & \oplus & \oplus & & \\
 & & \text{①} & C_m(X-Y) \arrow[r, "1"'] \arrow[d, "f_{m-1}"'] \arrow[dr, phantom, "\text{②}"] & C_m(X-Y) \arrow[d, "f_{m-2}"'] & \text{③} & \\
C': & 0 \arrow[r] & C_m(Y) \arrow[r, "\Delta"] & C_{m-1} \arrow[r, "\begin{pmatrix} d \\ g\Gamma \end{pmatrix}"'] & C_{m-2}\oplus C_m(X-Y) \arrow[r, "(d\ \ 0)"'] & C_{m-3} \arrow[r] & \cdots \\
 & & \oplus & \oplus & & & \\
 & & C_m(X-Y) \arrow[r, "1"'] & C_m(X-Y) & & &
\end{tikzcd}$$

$$f_m = \begin{pmatrix} 1 & 0 \\ g\Gamma di & 1 \end{pmatrix} : C_m = C_m(Y) \oplus C_m(X-Y) \longrightarrow C_m(Y) \oplus C_m(X-Y)$$

2ε-simple isom.

$$f_{m-1} = \begin{pmatrix} 1 & -dj \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ g\Gamma & 1 \end{pmatrix} : C_{m-1} \oplus C_m(X-Y) \longrightarrow C_{m-1} \oplus C_m(X-Y)$$

$$= \begin{pmatrix} 1 - djg\Gamma & -dj \\ g\Gamma & 1 \end{pmatrix}$$ 2ε-simple isom.

$$f_{m-2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$: 0-simple isom.

可換性

$$\text{①}\quad \urcorner\!\downarrow\ = \begin{pmatrix} 1 - djg\Gamma & -dj \\ g\Gamma & 1 \end{pmatrix} \begin{pmatrix} di & dj \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \overset{\Delta}{\overset{\shortparallel}{\overbrace{di - djg\Gamma di}}} & dj - dj\overset{\overset{1}{\wr^{2\varepsilon}}}{\overline{g\Gamma dj}} \\ g\Gamma di & \boxed{g\Gamma dj} \sim_{2\varepsilon} 1 \end{pmatrix}$$

$$\llcorner\!\!\rightarrow\ = \begin{pmatrix} \Delta & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ g\Gamma di & 1 \end{pmatrix} = \begin{pmatrix} \Delta & 0 \\ g\Gamma di & 1 \end{pmatrix}$$

$$\therefore\ \urcorner\!\downarrow\ \sim_{3\varepsilon}\ \llcorner\!\!\rightarrow$$

$$\text{②}\quad \urcorner\!\downarrow\ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & -1 \end{pmatrix}$$

$$\llcorner\!\!\rightarrow\ = \begin{pmatrix} d & 0 \\ g\Gamma & 0 \end{pmatrix} \begin{pmatrix} 1 - djg\Gamma & -dj \\ g\Gamma & 1 \end{pmatrix} = \begin{pmatrix} d - \overset{\overset{0}{\wr^{2\varepsilon}}}{\overline{dd}}jg\Gamma & -ddj \\ g\Gamma - \underbrace{g\Gamma dj}g\Gamma & -g\Gamma dj \end{pmatrix}$$

$$\therefore\ \urcorner\!\downarrow\ \sim_{3\varepsilon}\ \llcorner\!\!\rightarrow$$

③ $↴ = (d \ \ 0)$

$$↳ = (d \ \ 0)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = (d \ \ 0) \qquad \therefore\ ↴ \sim_{\varepsilon} ↳ .$$

よって. $\{f_r\}$ は 3ε chain map

逆は.

$$f_m^{-1} = \begin{pmatrix}1 & 0\\ -g\Gamma di & 1\end{pmatrix}$$

$$f_{m-1}^{-1} = \begin{pmatrix}1 & 0\\ -g\Gamma & 1\end{pmatrix}\begin{pmatrix}1 & dj\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & dj\\ -g\Gamma & -g\Gamma dj + 1\end{pmatrix}$$

$$f_{m-2}^{-1} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

① $$↱ = \begin{pmatrix}di & dj\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ -g\Gamma di & 1\end{pmatrix} = \begin{pmatrix}\overset{\triangle}{\overset{\shortparallel}{\overbrace{di - djg\Gamma di}}} & dj\\ 0 & 0\end{pmatrix}$$

$$⬏ = \begin{pmatrix}1 & dj\\ -g\Gamma & -g\Gamma dj+1\end{pmatrix}\begin{pmatrix}\triangle & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\triangle & dj\\ \underbrace{-g\Gamma\triangle}_{\shortparallel} & -g\Gamma dj+1\end{pmatrix} \underset{=}{\sim_{4\varepsilon}} \begin{pmatrix}\triangle & dj\\ 0 & 1\end{pmatrix}$$

$$-g\Gamma di + \underbrace{g\Gamma dj}_{\underset{1}{\wr}} g\Gamma di$$

② $$↱ = \begin{pmatrix}d & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & dj\\ -g\Gamma & -g\Gamma dj+1\end{pmatrix} = \begin{pmatrix}d & ddj\\ -g\Gamma & -g\Gamma dj+1\end{pmatrix} \sim_{2\varepsilon} \begin{pmatrix}d & 0\\ -g\Gamma & 0\end{pmatrix}$$

$$⬏ = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}d & 0\\ g\Gamma & 0\end{pmatrix} = \begin{pmatrix}d & 0\\ -g\Gamma & 0\end{pmatrix} \qquad \therefore\ ↱ \sim_{2\varepsilon} ⬏$$

③ $$↱ = (d \ \ 0)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = (d \ \ 0) = ⬏ .$$

∴ $\{f_r^{-1}\}$ は. 4ε chain map .

$$(d\ 0)\begin{pmatrix} d \\ \Gamma \end{pmatrix} = d^2 \underset{2\varepsilon}{\sim} 0 , \cdots$$

free
ε chain complex

1. $\hat{C}$:
$$\begin{tikzcd} {} \arrow[r] & 0 \arrow[r] & C_{m-1} \arrow[r, "\binom{d}{\Gamma}"] & C_{m-2} \oplus C_m \arrow[l, bend left] \arrow[r, "(d\ 0)"] & C_{m-3} \arrow[l, bend left] \arrow[r, "d"] & \cdots \arrow[r, "d"] & C_0 \arrow[r] & 0 \end{tikzcd}$$

$$\hat{\Gamma}_{m-2} = (\Gamma_{m-2}\ \ d), \quad \hat{\Gamma}_{m-3} = \begin{pmatrix} \Gamma_{m-3} \\ 0 \end{pmatrix}, \quad \hat{\Gamma}_r = \Gamma_r .$$

: すべて radius ε.

$$\hat{d}\hat{\Gamma} + \hat{\Gamma}\hat{d}$$

(m-1) $$(\Gamma_{m-2}\ \ d)\begin{pmatrix} d_{m-1} \\ \Gamma \end{pmatrix} = \overbrace{\Gamma_{m-2}d_{m-1}}^{\text{どこでも}2\varepsilon} + \overbrace{d\Gamma}^{2\varepsilon} \underset{2\varepsilon}{\sim} 1 . \text{ over } X-Y .$$

(m-2) $$\begin{pmatrix} d \\ \Gamma \end{pmatrix}(\Gamma\ \ d) + \begin{pmatrix} \Gamma \\ 0 \end{pmatrix}(d\ \ 0) = \overbrace{\begin{pmatrix} d\Gamma & d^2 \\ \Gamma^2 & \Gamma d \end{pmatrix}}^{\text{どこでも}2\varepsilon} + \overbrace{\begin{pmatrix} \Gamma d & 0 \\ 0 & 0 \end{pmatrix}}^{\text{どこでも}2\varepsilon}$$

$$= \begin{pmatrix} d\Gamma + \Gamma d & \boxed{d^2}\underset{2\varepsilon}{\sim} 0 . \\ \underset{\underset{2\varepsilon}{\sim} 0}{\boxed{\Gamma^2}} & \Gamma d \end{pmatrix} \underset{2\varepsilon}{\sim} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \underline{\underline{\text{over } X-Y}} .$$

(m-3) $$(d\ \ 0)\begin{pmatrix} \Gamma \\ 0 \end{pmatrix} + \Gamma d = \overbrace{d\Gamma + \Gamma d}^{\text{どこでも}2\varepsilon .} \underset{2\varepsilon}{\sim} 1 \text{ over } X-Y .$$

あとは自明.

∴ ε chain contraction over X−Y である.

$$\hat{\Gamma}^2 = ?$$

$$\hat{\Gamma}_{m-2}\hat{\Gamma}_{m-3} = (\Gamma_{m-2}\ \ d)\begin{pmatrix} \Gamma_{m-3} \\ 0 \end{pmatrix} = \Gamma^2 \underset{2\varepsilon}{\sim} 0 \quad \text{over } X-Y .$$

$$\hat{\Gamma}_{m-3}\hat{\Gamma}_{m-4} = \begin{pmatrix} \Gamma_{m-3} \\ 0 \end{pmatrix}\Gamma_{m-4} = \begin{pmatrix} \Gamma_{m-3}\Gamma_{m-4} \\ 0 \end{pmatrix} \underset{2\varepsilon}{\sim} 0 \quad \text{over } X-Y .$$

---.

∴ $\hat{\Gamma}$ は strong ε chain contraction over X−Y.

2. $\hat{C}_r(X-Y) = C'_r(X-Y)$ for all r.

3. $\hat{d}_r | X-Y^{\varepsilon} = d'_r | X-Y^{\varepsilon} : C_r(X-Y^{\varepsilon}) \to C_r(X-Y)$

C' は 3ε free chain complex.

By 2.2 $\delta = \varepsilon$, $\gamma = 3\varepsilon$, $\varepsilon' = 3\varepsilon$.

$C' \oplus \Sigma\hat{C}$ is $(n, X - Y^{\overbrace{7\varepsilon + 4\varepsilon + 6\varepsilon}^{17\varepsilon}})$-stably $\overbrace{3\varepsilon + 3\varepsilon + 3\varepsilon}^{9\varepsilon}$ equivalent to 0.

同様に $\delta = \varepsilon$, $\gamma = \varepsilon$, $\varepsilon' = \varepsilon$.

$\hat{C} \oplus \Sigma\hat{C}$ は $(n, X - Y^{\underbrace{7\varepsilon + 4\varepsilon + 2\varepsilon}_{13\varepsilon}})$-stably $\underbrace{3\varepsilon + 3\varepsilon + \varepsilon}_{7\varepsilon}$ equivalent to 0

すでにみたように C と C' は n-stably 4ε equivalent

$$C = C \oplus 0 \overset{(n, X-Y^{13\varepsilon})}{\underset{7\varepsilon}{\sim}} C' \oplus \Sigma\hat{C} \oplus \hat{C} \overset{n, X-Y^{17\varepsilon}}{\underset{9\varepsilon}{\sim}} 0 \oplus \hat{C} = \hat{C}.$$

$$\therefore\ C \overset{n, X-Y^{17\varepsilon}}{\underset{16\varepsilon}{\sim}} \hat{C}.$$

□

この応用では、δは実質的に使っていない。
δをεでおきかえてもよいか？

Corollary 2.4. (additive inverse for $n > 1$)

$$C \oplus \Sigma\hat{C} \overset{n, X-Y^{17\varepsilon}}{\underset{16\varepsilon}{\sim}} \hat{C} \oplus \Sigma\hat{C} \overset{n, X-Y^{13\varepsilon}}{\underset{7\varepsilon}{\sim}} 0$$

$$\therefore\ C \oplus \Sigma\hat{C} \overset{n, X-Y^{17\varepsilon}}{\underset{23\varepsilon}{\sim}} 0.$$

existence of inverse , $n=1$.

Lemma 2.2′ Let $C: 0 \to C_1 \xrightarrow{d} C_0 \to 0$ be a 1-dimensional free ε chain complex and $C': 0 \to C_1' \xrightarrow{d'} C_0' \to 0$ be a 1-dimensional free ε' chain complex. Assume

1. C has an ε chain contraction Γ over $X - W^{\varepsilon}$,
2. $C_1'(X-W) = C_0(X-W)$, $C_0'(X-W) = C_1(X-W)$,
3. $d' | X - W^{\varepsilon} \sim_{\varepsilon} \Gamma | X - W^{\varepsilon} : C_1'(X-W^{\varepsilon}) \to C_0'(X-W)$.

Let $\gamma = \max\{\varepsilon, \varepsilon'\}$. Then the direct sum $C \oplus C'$ is $(5\varepsilon+\gamma)$-simply isomorphic to the direct sum of a 1-dimensional free $3\varepsilon+\gamma$ chain complex on $P_{W^{5\varepsilon+\gamma}}$ and a 1-dimensional trivial chain complex on P_X.

Proof:

$$\tilde{\alpha} = \begin{cases} d & \text{over } X - W^{\varepsilon} \\ 0 & \text{over } W^{\varepsilon} \end{cases} : C_0' \longrightarrow C_0 \qquad ⓔ$$

$$\tilde{\Gamma} = \begin{cases} \Gamma & \text{over } X - W^{\varepsilon} \\ 0 & \text{over } W^{\varepsilon} \end{cases} : C_1' \longrightarrow C_1 \qquad ⓔ$$

$$\hat{\Gamma} = \begin{cases} \Gamma & \text{over } X - W^{\varepsilon} \\ 0 & \text{over } W^{\varepsilon} \end{cases} : C_0 \longrightarrow C_0' \qquad ⓔ$$

とおく。

(2の条件より well-defined)

$C \oplus C'$ と次の E の stable isom. をつくる.

$E_1 = C_1' \oplus C_1$, $E_0 = C_0 \oplus C_0'$

$$d_E = \begin{cases} \begin{pmatrix} d\tilde{\Gamma} - \underbrace{\tilde{\alpha}\hat{\Gamma}d\tilde{\Gamma}}_{4\varepsilon} + \underbrace{\tilde{\alpha}d'}_{\varepsilon+\varepsilon'} & d - \tilde{\alpha}\hat{\Gamma}d \\ \hat{\Gamma}d\tilde{\Gamma} & \hat{\Gamma}d \end{pmatrix} & \text{over } W^{2\varepsilon} \\ 1 & \text{over } X - W^{2\varepsilon} \end{cases}$$

$\max\{4\varepsilon, \varepsilon+\varepsilon'\}$ morphism.

E は free(〃) chain complex.

$$f_1 = \begin{pmatrix} 0 & 1 \\ 1 & -\tilde{\Gamma} \end{pmatrix} : C_1 \oplus C_1' \longrightarrow C_1' \oplus C_1 = E_1$$

ε-simple iso.

$$f_0 = \begin{pmatrix} 1 & -\tilde{\alpha} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \hat{\Gamma} & -1 \end{pmatrix} : C_0 \oplus C_0' \longrightarrow C_0 \oplus C_0' = E_0$$

2ε-simple iso.

㊟ $\begin{pmatrix} 1 & 0 \\ \hat{\Gamma} & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \hat{\Gamma} & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \hat{\Gamma}-\hat{\Gamma} & 1 \end{pmatrix}$

~ 1.

主張 $d_E \sim_{\max\{4\varepsilon,\, \varepsilon+\varepsilon'\}} \underbrace{f_0 (d \oplus d') f_1^{-1}}$

sizeはきちんとみると

$\max\{4\varepsilon, \varepsilon+\varepsilon'\}$

粗にみると. $2\varepsilon + \varepsilon + \max\{\varepsilon, \varepsilon'\}$

$= 3\varepsilon + \max\{\varepsilon, \varepsilon'\}$

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] \\
C_1 \oplus C_1' \arrow[d, "d \oplus d'"] \arrow[rr, shift left, "f_1"] & & C_1' \oplus C_1 = E_1 \arrow[ll, shift left, "f_1^{-1}"] \arrow[d] \\
C_0 \oplus C_0' \arrow[d] \arrow[rr, shift left, "f_0"] & & C_0' \oplus C_0 = E_0 \arrow[ll, shift left, "f_0^{-1}"] \arrow[d] \\
0 & & 0
\end{tikzcd}$$

$$f_1 = \begin{pmatrix} 0 & 1 \\ 1 & -\tilde{\Gamma} \end{pmatrix}, \quad f_1^{-1} = \begin{pmatrix} \tilde{\Gamma} & 1 \\ 1 & 0 \end{pmatrix}$$

$$f_0 = \begin{pmatrix} 1 - \tilde{\alpha}\hat{\Gamma} & \tilde{\alpha} \\ \hat{\Gamma} & -1 \end{pmatrix}$$

$$f_0^{-1} = \begin{pmatrix} 1 & 0 \\ \hat{\Gamma} & -1 \end{pmatrix}\begin{pmatrix} 1 & \tilde{\alpha} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \tilde{\alpha} \\ \hat{\Gamma} & \hat{\Gamma}\tilde{\alpha} - 1 \end{pmatrix}$$

$f_0\,(d\oplus d')\,f_1^{-1}$

$$= \begin{pmatrix} 1-\tilde{\alpha}\hat{\Gamma} & \tilde{\alpha} \\ \hat{\Gamma} & -1 \end{pmatrix}\begin{pmatrix} d & 0 \\ 0 & d' \end{pmatrix}\begin{pmatrix} \tilde{\Gamma} & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\tilde{\alpha}\hat{\Gamma} & \tilde{\alpha} \\ \hat{\Gamma} & -1 \end{pmatrix}\begin{pmatrix} d\tilde{\Gamma} & d \\ d' & 0 \end{pmatrix} = \left(\begin{array}{c:c} d\tilde{\Gamma}-\tilde{\alpha}\hat{\Gamma}d\tilde{\Gamma}+\tilde{\alpha}d' & d-\tilde{\alpha}\hat{\Gamma}d \\ \hat{\Gamma}d\tilde{\Gamma}-d' & \hat{\Gamma}d \end{array}\right)$$

(1,1)成分 $= d\boxed{\tilde{\Gamma}} - \tilde{\alpha}\hat{\Gamma}d\boxed{\tilde{\Gamma}} + \tilde{\alpha}\boxed{d'}$

$\underset{4\varepsilon}{\sim} d\Gamma - \tilde{\alpha}\hat{\Gamma}\underbrace{\boxed{d\Gamma}}_{\underset{2\varepsilon}{\wr}\ 1} + \tilde{\alpha}\Gamma$

$\underset{4\varepsilon}{\sim} d\Gamma - \tilde{\alpha}\underset{\overset{\shortparallel}{\Gamma}}{\boxed{\hat{\Gamma}}} + \tilde{\alpha}\Gamma = \qquad d\Gamma - \tilde{\alpha}\Gamma + \tilde{\alpha}\Gamma$

$\underset{2\varepsilon}{\sim} d\Gamma \underset{2\varepsilon}{\sim} 1$

（すべて $X-W^{\varepsilon}$ 上で ）.

(1,2) 成分 $= d - \tilde{\alpha}\hat{\Gamma}d$ 　　: $\underline{X-W^{2\varepsilon}}$ 上で

$= d - \tilde{\alpha}\boxed{\Gamma d}$

$\underset{3\varepsilon}{\sim} d - \tilde{\alpha} \underset{\varepsilon}{\sim} 0$

(2,1) 成分 $= \hat{\Gamma}d\boxed{\tilde{\Gamma}} - \boxed{d'}$ 　　$\underline{X-W^{\varepsilon}}$ 上で.

$\underset{3\varepsilon}{\sim} \hat{\Gamma}\boxed{d\Gamma} - \Gamma$

$\underset{3\varepsilon}{\sim} \hat{\Gamma} - \Gamma \underset{\varepsilon}{\sim} 0$

(2,2) 成分 $= \hat{\Gamma}d$ 　　$\underline{X-W^{2\varepsilon}}$ 上で

$= \Gamma d \underset{2\varepsilon}{\sim} 1$

$\therefore\ f_0(d\oplus d')f_1' \,|\, X-W^{2\varepsilon} \underset{4\varepsilon}{\sim} 1.$

よて. ⤵ $= d_E f_1 \sim_{\max\{5\varepsilon,\, 2\varepsilon+\varepsilon'\}} \overset{2\varepsilon}{\frown}{f_0}\,\overset{\max\{\varepsilon,\varepsilon'\}}{(d\oplus d')}\,\overset{\overset{1}{\sim_{2\varepsilon}}}{f_1^{-1} f_1}$

$\sim_{4\varepsilon+\max\{\varepsilon,\varepsilon'\}} \quad f_0(d\oplus d') =$ ↳ .

$\wedge$

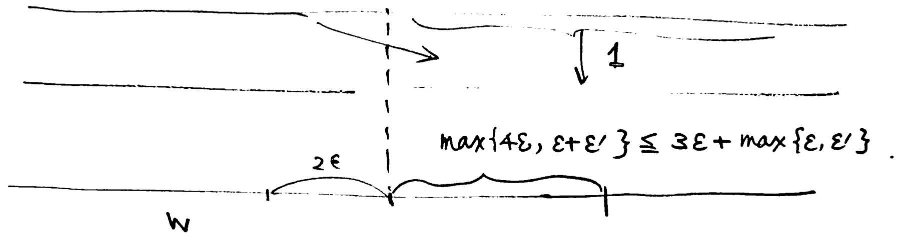
∴ → は $4\varepsilon+\max\{\varepsilon,\varepsilon'\}$ chain map .

↲ $= \overset{2\varepsilon}{f_0^{-1}}\, d_E \sim_{\max\{6\varepsilon,\, 3\varepsilon+\varepsilon'\}} \overset{\overset{1}{\sim_{4\varepsilon}}}{f_0^{-1} f_0}\,\overset{\max\{\varepsilon,\varepsilon'\}}{(d\oplus d')}\,\overset{\varepsilon}{f_1^{-1}}$

$\sim_{5\varepsilon+\max\{\varepsilon,\varepsilon'\}} \quad (d\oplus d') f_1^{-1} =$ ↲ .

∴ ← は $5\varepsilon+\max\{\varepsilon,\varepsilon'\}$ chain map .

∴ $f = \{f_r\} : C\oplus C' \longrightarrow E$ は. $(5\varepsilon+\max\{\varepsilon,\varepsilon'\})$-simple isom.



$E = E(W^{5\varepsilon+\max\{\varepsilon,\varepsilon'\}}) \oplus (\text{trivial})$. □

---

additive inverse $[C] \in Wh(X, Y, p_X, \varepsilon)$.

$[C' : 0 \to C_1 \xrightarrow{\Gamma} C_0 \to 0]$ が additive inverse.

$\varepsilon' = \varepsilon$

$C\oplus C' \underset{6\varepsilon}{\overset{(1,\, X - W^{5\varepsilon+\delta})}{\smile}} 0$ 6ε-simple iso to (triv) ⊕ $\sqcup_{\text{on } P_W^{6\varepsilon}}$

2.1 証明終わり □

$[C] = [C'] \in Wh(X, Y, p_X, n, \varepsilon)$ とする。

$$C = C^{(1)} \sim C^{(2)} \sim \cdots \sim C^{(m)} = C'$$

$\sim$ は $\underset{40\varepsilon}{\overset{(n,\, X-Y^{20\varepsilon})}{\sim}}$.

Corollary 2.4 により
(n>1) $\quad C^{(i)} \oplus \widehat{\Sigma C^{(i)}} \underset{23\varepsilon}{\overset{(n,\, X-Y^{17\varepsilon})}{\sim}} 0$.

Lemma 2.2′ により
(n=1) $\quad C^{(i)} \oplus (C^{(i)})' \underset{6\varepsilon}{\overset{(1,\, X-Y^{6\varepsilon})}{\sim}} 0$.

これらをまとめて $D^{(i)}$ とかこう.

$$C^{(i)} \oplus {}^{\exists}D^{(i)} \underset{23\varepsilon}{\overset{(n,\, X-Y^{17\varepsilon})}{\sim}} 0 .$$

$$C \oplus 0 \underset{23\varepsilon}{\overset{(n,\, X-Y^{17\varepsilon})}{\sim}} C \oplus (C^{(1)} \oplus D^{(1)}) \oplus (C^{(2)} \oplus D^{(2)}) \oplus \cdots \oplus (C^{(n)} \oplus D^{(n)})$$

$$\underset{40\varepsilon}{\overset{(n,\, X-Y^{20\varepsilon})}{\sim}} C \oplus (C^{(2)} \oplus D^{(1)}) \oplus (C^{(3)} \oplus D^{(2)}) \oplus \cdots \oplus (C^{(n)} \oplus D^{(n-1)}) \oplus (C^{(n)} \oplus D^{(n)}$$

$$= (C^{(1)} \oplus D^{(1)}) \oplus \cdots \oplus (C^{(n)} \oplus D^{(n)}) \oplus C^{(n)}$$

$$\underset{23\varepsilon}{\overset{(n,\, X-Y^{17\varepsilon})}{\sim}} C'$$

$$\therefore C \underset{86\varepsilon}{\overset{(n,\, X-Y^{20\varepsilon})}{\sim}} C'$$

Proposition 2.5

$n > 1$.

$[C, d]$, $[C', d'] \in WR(X, Y, p_X, n, \varepsilon)$

$$\left.\begin{array}{l} C_r(X-Y) = C'_r(X-Y) \\ d_r | X-Y^{\varepsilon} = d'_r | X-Y^{\varepsilon} \end{array}\right\} \forall r.$$

Γ : C の strong ε chain contraction over $X-Y$

$\Gamma' : C'_r \longrightarrow C'_{r+1}$ を $\Gamma' = \begin{cases} \Gamma & \text{over } X-Y^{\varepsilon} \\ 0 & \text{over } Y^{\varepsilon} \end{cases}$ と定める.

: radius ε.

over $X-Y^{2\varepsilon}$

$$d'\Gamma' + \Gamma'd' = d\Gamma + \Gamma d \underset{2\varepsilon}{\sim} 1.$$

$$\Gamma'^2 = \Gamma^2 \underset{2\varepsilon}{\sim} 0.$$

$\therefore$ Γ' は strong ε chain contraction over $X-Y^{2\varepsilon}$. of C'.

Lemma 2.3 により.

$$C \underset{16\varepsilon}{\overset{(n,\, X-Y^{17\varepsilon})}{\sim}} \underset{\substack{\uparrow \\ (n-1)\text{-dim}}}{\hat{C}}, \qquad C' \underset{16\varepsilon}{\overset{(n,\, X-Y^{19\varepsilon})}{\sim}} \hat{C}'$$

Lemma 2.2 により.

$$\hat{C} \oplus \Sigma\hat{C} \underset{7\varepsilon}{\overset{(n,\, X-Y^{13\varepsilon})}{\sim}} 0$$

$\Sigma\hat{C}$ has an ε chain contraction over $X-Y$ $\therefore$ ($X-Y^{\varepsilon}$ で OK)

$$\hat{C}' \oplus \Sigma\hat{C} \underset{7\varepsilon}{\overset{(n,\, X-Y^{13\varepsilon})}{\sim}} 0$$

これらより

$$C \underset{16\varepsilon}{\overset{(n,\, X-Y^{17\varepsilon})}{\sim}} \hat{C} \oplus (\underbrace{\Sigma\hat{C} \oplus}_{\underset{0}{\wr}} \underset{\underset{C'}{\wr}}{\hat{C}'})$$

$$\underset{16\varepsilon}{\overset{(n,\, X-Y^{19\varepsilon})}{\sim}} C'$$

$$\therefore C \underset{32\varepsilon}{\overset{(n,\, X-Y^{19\varepsilon})}{\sim}} C'$$

$$\hat{C} : \quad 0 \longrightarrow C_{n-1} \xrightarrow{\binom{d}{\Gamma}} C_{n-2} \oplus C_n \xrightarrow{(d\ 0)} C_{n-3} \xrightarrow{d} \cdots \xrightarrow{d} C_0 \longrightarrow 0$$

$$\hat{C}' : \quad 0 \longrightarrow C'_{n-1} \xrightarrow[\binom{d}{\Gamma'}]{} C'_{n-2} \oplus C'_n \xrightarrow[(d'\ 0)]{} C'_{n-3} \xrightarrow[d']{} \cdots \xrightarrow[d']{} C'_0 \longrightarrow 0$$

$X-Y^{\varepsilon}$ で boundary maps が一致. (2.2の(3)✓ (2)も✓)

φ が $\varphi_* : Wh(X, Y, p_X, \delta) \longrightarrow Wh(X', Y', p_{X'}, \varepsilon)$ を induce するための条件

1. δ chain complex $\rightsquigarrow$ ε chain complex
2. strongly δ contractible over $X - Y$ $\rightsquigarrow$ strongly ε contractible over $X' - Y'$.
3. $(n, X - Y^{20\delta})$-stable 40δ equivalence
   $\rightsquigarrow$ $(n, X' - Y'^{20\varepsilon})$-stable 40ε equivalence.

ならばよい。

1. radius δ $\longrightarrow$ radius ε. $C(\delta, \varepsilon, 1)$

   $d^2 \underset{2\delta}{\sim} 0$ $\longrightarrow$ $d^2 \underset{2\varepsilon}{\sim} 0$. $C(\delta, \varepsilon, 2)$

2. Γ : radius δ $\longrightarrow$ radius ε $C(\delta, \varepsilon, 1)$

   $d\Gamma, \Gamma d$ 2δ $\longrightarrow$ 2ε $C(\delta, \varepsilon, 2)$

   $d\Gamma + \Gamma d \underset{2\delta}{\sim} 1$ over $X - Y$ $\longrightarrow$ $\underset{2\varepsilon}{\sim}$ over $X' - Y'$ $\bar{\varphi}(Y) \subset Y'$

   $C(\delta, \varepsilon, 2)$

   $\Gamma^2 \underset{2\delta}{\sim} 0$ over $X - Y$. $\underset{2\varepsilon}{\sim}$. $C(\delta, \varepsilon, 2)$ $\bar{\varphi}(Y) \subset Y'$

3. $\underline{\bar{\varphi}(Y^{20\delta}) \subset (Y')^{20\varepsilon}}$. が必要.

   直接これを仮定. 又は $\bar{\varphi}(Y) \subset Y'$ かつ $C(\delta, \varepsilon, 20)$

radius 40δ $\longrightarrow$ 40ε $C(\delta, \varepsilon, 40)$

chain map だから.

$\underset{40\delta}{\sim}$ $\longrightarrow$ $\underset{40\varepsilon}{\sim}$ $C(\delta, \varepsilon, 40)$

40δ-simple iso $\longrightarrow$ 40ε-simple iso $C(\delta, \varepsilon, 40)$

$\therefore$ $C(\delta, \varepsilon, n)$, $n = 1, 2, 40$, および $\bar{\varphi}(Y) \subset Y'$

および ($\bar{\varphi}(Y^{20\delta}) \subset (Y')^{20\varepsilon}$ 又は $C(\delta, \varepsilon, 20)$)

Proposition 2-7.

$C_\Gamma : C_{odd} = C_1 \oplus C_3 \oplus \cdots \to C_{even} = C_0 \oplus C_2 \oplus \cdots$

$\tau : [C] \longmapsto [C_\Gamma]$ well-defined.

$Wh(X, Y, p_X, n, \varepsilon) \longrightarrow Wh(X, Y \quad , p_X, \quad)$

Proof:

① Γのとり方によらないこと。

Γ, Γ' : 2つの strong ε chain contraction over X−Y of C.

$\underbrace{(1+\Gamma'\Gamma)}_{C_{even}\to C_{even}}(d+\Gamma) \sim_{3\varepsilon} (d+\Gamma')(\underbrace{1+\Gamma'\Gamma}_{C_{odd}\to C_{odd}}) : C_{odd} \to C_{even}$ $\underline{\text{over } X-Y^{\varepsilon}}$.

$(\because)\ (1+\Gamma'\Gamma)(d+\Gamma) = d+\Gamma+\Gamma'\Gamma d + \Gamma'\underbrace{\Gamma\Gamma}$

$\sim_{3\varepsilon} d+\Gamma+\Gamma'(1-d\Gamma) = d+\Gamma+\Gamma'-\Gamma' d\Gamma$ over X−Y.

$(d+\Gamma')(1+\Gamma'\Gamma) = d+d\Gamma'\Gamma+\Gamma'+\Gamma'\Gamma'\Gamma$

$\sim_{3\varepsilon} d+(1-\Gamma' d)\Gamma+\Gamma'$ over $X-Y^{\varepsilon}$

$= d+\Gamma-\Gamma' d\Gamma+\Gamma'$

$1+\Gamma'\Gamma : C_{odd} \to C_{odd}$ は

$(1+\Gamma'\Gamma) = (1+\Gamma'\Gamma|C_1)(1+\Gamma'\Gamma|C_3)\cdots$ とかける.

even も同様.

$\therefore\ 1+\Gamma'\Gamma : C_{odd} \to C_{odd}$, $1+\Gamma'\Gamma : C_{even} \to C_{even}$ は nε-simple iso's.

$\Delta = \underbrace{(1+\Gamma'\Gamma)}_{2\varepsilon}\underbrace{(d+\Gamma)}_{\varepsilon}\underbrace{(1+\Gamma'\Gamma)^{-1}}_{n\varepsilon} : C_{odd} \to C_{even}$ とおく.

(n+3)ε morphism.

$$\begin{tikzcd} C_{odd} \arrow[r, "\Delta"] & C_{even} \arrow[l, bend left, dashed, "\hbar"] \end{tikzcd}$$

$*\ (d+\Gamma)(d+\Gamma)$

$= d^2 + d\Gamma + \Gamma d + d^2 \underset{2\varepsilon}{\sim} 1$

over $X - Y$.

$\hbar = (1+\Gamma'\Gamma)(d+\Gamma)(1+\Gamma'\Gamma)^{-1}$.

: radius $(n+3)\varepsilon$.

$$\Delta\hbar = \underset{2\varepsilon}{(1+\Gamma'\Gamma)}\underset{\varepsilon}{(d+\Gamma)}\underset{\overset{\wr}{1}}{\underbrace{\boxed{(1+\Gamma'\Gamma)^{-1}(1+\Gamma'\Gamma)}}_{2n\varepsilon}}\underset{\varepsilon}{(d+\Gamma)}\underset{2\varepsilon}{(1+\Gamma'\Gamma)^{-1}}.$$

← 雑.

$$\underset{(2n+6)\varepsilon}{\sim} \underset{2}{(1+\Gamma'\Gamma)}\underbrace{(d+\Gamma)(d+\Gamma)}_{\wr\ 2\varepsilon}\underbrace{(1+\Gamma'\Gamma)^{-1}}_{n\varepsilon}.$$

$$\underset{(n+4)\varepsilon}{\sim} (1+\Gamma'\Gamma)(1+\Gamma'\Gamma)^{-1} \quad \text{over } \underline{X - Y^{n\varepsilon}}.$$

$$\underset{2n\varepsilon}{\sim} 1$$

$(2n+6)\varepsilon = 2(n+3)\varepsilon$.

$\hbar\Delta$ = 式は全く同じ.

$\therefore\ [\Delta] \in W\hbar(X, Y^{n\varepsilon}, p_X, (n+3)\varepsilon)$

$$\begin{tikzcd} C_\Gamma & \Delta \\ 0 \arrow[d] & 0 \arrow[d] \\ C_{odd} \arrow[r, shift left, "1+\Gamma'\Gamma"] \arrow[d, "d+\Gamma"'] & C_{odd} \arrow[l, shift left, "(1+\Gamma'\Gamma)^{-1}"] \arrow[d, "\Delta"] \\ C_{even} \arrow[r, shift left, "1+\Gamma'\Gamma"] \arrow[d] & C_{even} \arrow[l, shift left, "(1+\Gamma'\Gamma)^{-1}"] \arrow[d] \\ 0 & 0 \end{tikzcd}$$

⤵ $= \Delta(1+\Gamma'\Gamma)$

$$= \underset{2\varepsilon}{(1+\Gamma'\Gamma)}\underset{\varepsilon}{(d+\Gamma)}\underset{\overset{\wr}{1}}{\underbrace{(1+\Gamma'\Gamma)^{-1}\cancel{(1+\Gamma'\Gamma)}}_{2n\varepsilon}}$$

$n\varepsilon$

雑!!

$\underset{(2n+3)\varepsilon}{\sim}$ ↳

$(n+3)\varepsilon$

$$↲ = \underset{\overset{\wr}{1}}{\underbrace{(1+\Gamma'\Gamma)^{-1}(1+\Gamma'\Gamma)}_{2n\varepsilon}}\underset{\varepsilon}{(d+\Gamma)}\underset{n\varepsilon}{(1+\Gamma'\Gamma)^{-1}}.$$

$n\varepsilon$

$\underset{(3n+1)\varepsilon}{\sim}$ ↲

$(2n+1)\varepsilon$

$\therefore\ C_\Gamma$ と Δ は $(3n+1)\varepsilon$-simple isomorphic.

$(2n+3)\varepsilon$ -

ここは「雑」でかまわない.

$$\begin{tikzcd}
\bar{C}_{\bar{\Gamma}} & & \hat{C}_{\hat{\Gamma}} \\
\bar{C}_{odd} \arrow[rr, shift left, "\overset{40\varepsilon\text{-simple}}{f}"] \arrow[d, "\bar{d}+\bar{\Gamma}"'] & & \hat{C}_{odd} \arrow[ll, shift left, "f^{-1}"] \arrow[d, "\hat{d}+\hat{\Gamma}'"] \\
\bar{C}_{even} \arrow[rr, shift left, "f"] & & \hat{C}_{even} \arrow[ll, shift left, "f^{-1}"]
\end{tikzcd}$$

$$↴ = (\hat{d}+\hat{\Gamma}')f = \underbrace{\hat{d}f}_{\substack{\wr_{40\varepsilon} \\ f\bar{d}}} + \underset{40\varepsilon\ \varepsilon}{f\bar{\Gamma}}\underbrace{f^{-1}f}_{80\varepsilon} \underset{121\varepsilon}{\sim} f\bar{d} + f\bar{\Gamma} = ↳$$

121ε chain map.

$$↲ = f^{-1}(\hat{d}+\hat{\Gamma}') = f^{-1}\hat{d} + \underbrace{f^{-1}f}_{80\varepsilon}\bar{\Gamma}f^{-1} \underset{121\varepsilon}{\sim} \bar{d}f^{-1} + \bar{\Gamma}f^{-1} = ⬐$$

← も 121ε chain map.

∴ $\bar{C}_{\bar{\Gamma}}$ と $\hat{C}_{\hat{\Gamma}'}$ は. 121ε-simple isomorphic.

$[C_{\Gamma}] = [\bar{C}_{\bar{\Gamma}}] \in Wh(X, Y^{20\varepsilon}, p_X, 40\varepsilon)$.

は明らか。(基底すべて $Y^{20\varepsilon}$ 上のみ).

Γ' を C' の任意の strong ε chain contraction over $X-Y$ とする。

$\Gamma'\oplus 0 = \hat{\Gamma}$ とかくと. $\hat{\Gamma}$ は $\hat{C}$ の strong ε chain contraction over $X-Y^{20\varepsilon}$.

40ε chain cx.

∴ $[C'_{\Gamma'}] = [\hat{C}_{\hat{\Gamma}}] \in Wh(X, Y^{20\varepsilon}, p_X, 40\varepsilon)$

$\hat{C}$ に対して. 前段落の議論をあてはめると.

$\hat{\Gamma}$ も $\hat{\Gamma}'$ も共に strong 81ε chain contraction over $X-Y^{60\varepsilon}$ だから.

㊟ $1+\Gamma'\Gamma$ について

$(1+\Gamma'\Gamma | C_1)(1+\Gamma'\Gamma | C_3)\cdots$

$(1+\Gamma'\Gamma | C_0)(1+\Gamma'\Gamma | C_2)\cdots$

$n = 2k$ のとき.

odd $1, 3, 5, \cdots, 2k-3$: $(k-1)$ 個

ひとつひとつが 2ε isomorphism.

∴ 合成は. $(k-1)2\varepsilon = (n-2)\varepsilon$-simple

even $0, 2, 4, \cdots, 2k-2$: k 個

$k \cdot 2\varepsilon = 2k\varepsilon = n\varepsilon$-simple

$n = 2k+1$ のとき

odd $1, 3, 5, \cdots, 2k-1$: k 個

$(k) \cdot 2\varepsilon = 2k\varepsilon = (n-1)\varepsilon$-simple

even $0, 2, 4, \cdots, 2k-2$: k 個

$2k\varepsilon = (n-1)\varepsilon$-simple

∴ $1+\Gamma'\Gamma$ は $n\varepsilon$-simple.

さらに.

$D = \Gamma'\Gamma$ とかく. : degree 2 :

$k = \left[\frac{n}{2}\right]$ とおく

$$\frac{n}{2} - 1 < k \leq \frac{n}{2}$$

$$n - 2 < 2k \leq n$$

$$n < 2k+2 \leq n+2 \qquad n < 2(k+1)$$

$$\therefore D^{k+1} = 0$$

$$(1+D)^{-1} = 1 - D + D^2 - D^3 + \cdots \pm D^k .$$

実際. $(1+D)(1-D+D^2-D^3+\cdots \pm D^k)$

$$= 1 - D + D^2 - D^3 + \cdots \pm D^k$$
$$+ D - D^2 + D^3 - \cdots \mp D^k \ \overset{\overset{0}{\shortparallel}}{\pm D^{k+1}}$$

$$\underset{\substack{2k\varepsilon \\ \wedge\shortparallel \\ n\varepsilon}}{\sim} 1 .$$

同様に $(1-D+\cdots \pm D^k)(1+D) \underset{n\varepsilon}{\sim} 1 .$

したがって. $1+D = 1+\Gamma'\Gamma$ は $n\varepsilon$ isomorphism であり. かつ さらに.

$$(1+D)(1+D)^{-1} \underset{n\varepsilon}{\sim} 1 , \quad (1+D)^{-1}(1+D) \underset{n\varepsilon}{\sim} 1 .$$

しかしこの $(1+D)^{-1}$ と. $1+D$ の $n\varepsilon$-simple isomorphism としての

逆 $\left.\begin{matrix} \cdots(1-\Gamma'\Gamma|C_3)(1-\Gamma'\Gamma|C_1) \\ \cdots(1-\Gamma'\Gamma|C_2)(1-\Gamma'\Gamma|C_0) \end{matrix}\right\}$ とは どういう関係にあるか?

明らかに一致する。

$$\therefore (1+\Gamma'\Gamma)(1+\Gamma'\Gamma)^{-1} \underset{n\varepsilon}{\sim} 1$$

$$(1+\Gamma'\Gamma)^{-1}(1+\Gamma'\Gamma) \underset{n\varepsilon}{\sim} 1 .$$

$[C], [C'] \in Wh(X, Y, p_X, n, \varepsilon)$.

仮定

$\exists f : \bar{C} = C \oplus D \longrightarrow \hat{C} = C' \oplus D'$

40ε-simple isomorphism.

D, D' : n-dimensional free 40ε chain cx's on $Y^{20\varepsilon}$.

Γ : a strong ε chain contraction over $X - Y$ of C.

$\bar\Gamma = \Gamma \oplus 0$ は 〃 〃 〃 〃 〃 $X - Y^{20\varepsilon}$ of $\underbrace{\bar{C}}_{40\varepsilon \text{ chain complex}}$.

主張 $\hat\Gamma' = \overset{40\varepsilon}{\widehat{f}}\,\overset{\varepsilon}{\widehat{\bar\Gamma}}\,\overset{40\varepsilon}{\widehat{f^{-1}}}$ は strong 81ε chain contraction over $X - Y^{60\varepsilon}$ of $\hat{C}$.

∵

$$\hat d\hat\Gamma' + \hat\Gamma'\hat d = \underbrace{\hat d f}_{\wr 40\varepsilon}\underset{\varepsilon}{\bar\Gamma}\underset{40\varepsilon}{f^{-1}} + \overset{40\varepsilon}{f}\overset{\varepsilon}{\bar\Gamma}\underbrace{f^{-1}\hat d}_{\wr 40\varepsilon} \sim_{81\varepsilon} f\bar d\bar\Gamma f^{-1} + f\bar\Gamma\bar d f^{-1}$$

$$= \underset{40\varepsilon}{f}(\underbrace{\bar d\bar\Gamma + \bar\Gamma\bar d}_{\substack{\wr\varepsilon \\ 1 \\ \text{over } X - Y^{20\varepsilon}}})\underbrace{f^{-1}}_{40\varepsilon} \underset{\substack{\uparrow 81\varepsilon \\ \text{over} \\ X - Y^{60\varepsilon}}}{\sim} f f^{-1} \sim_{80\varepsilon} 1.$$

∴ $\hat\Gamma$ は 81ε chain contraction over $X - Y^{60\varepsilon}$.

$\hat d\hat\Gamma'$, $\hat\Gamma'\hat d$ a radius は 共に $\leq 162\varepsilon$.

$$\hat\Gamma'^2 = \underset{40\varepsilon}{f}\,\underset{\varepsilon}{\bar\Gamma}\underbrace{f^{-1}f}_{\substack{\wr 80\varepsilon \\ 1}}\underset{\varepsilon}{\bar\Gamma}\,\underset{40\varepsilon}{f^{-1}} \sim_{162\varepsilon} \underset{40\varepsilon}{f}\underbrace{\bar\Gamma^2}_{\substack{\wr 2\varepsilon \\ 0}}\underset{40\varepsilon}{f^{-1}} \underset{\substack{82\varepsilon \\ \uparrow \\ \text{over } X - Y^{60\varepsilon}}}{\sim} 0$$

∴ $\hat\Gamma'^2 \sim_{162\varepsilon} 0$ over $X - Y^{60\varepsilon}$

∴ strong 81ε chain contraction over $X - Y^{60\varepsilon}$.

$[C_\Gamma] \in Wh(X, Y, p_X, n, \varepsilon)$

$[C_\Gamma] = [\Delta] \in Wh(X, Y^{n\varepsilon}, p_X, n, (n+3)\varepsilon)$

$\left[\because \quad 40(n+3)\varepsilon > (3n+1)\varepsilon. \right]$.

次に $[\Delta]$ と $[C_{\Gamma'}]$ の比較.

$$\Delta = \underbrace{(1+\Gamma'\Gamma)(d+\Gamma)}_{\wr_{3\varepsilon}} \underbrace{(1+\Gamma'\Gamma)^{-1}}_{n\varepsilon} \underset{(n+3)\varepsilon}{\sim} \underset{\varepsilon}{(d+\Gamma')} \overbrace{(1+\Gamma'\Gamma)(1+\Gamma'\Gamma)^{-1}}^{\overset{1}{\wr^{n\varepsilon}\ \wr^{2n\varepsilon}}}$$

$(d+\Gamma')(1+\Gamma'\Gamma)$

over $X - Y^{\varepsilon}$.

over $X - Y^{(n+1)\varepsilon}$.

$\underset{\substack{(2n+1)\varepsilon \\ (n+1)\varepsilon}}{\sim} d+\Gamma'$

$\therefore \ \Delta \underset{\substack{(2n+3)\varepsilon \\ (n+3)\varepsilon}}{\sim} d+\Gamma'$ over $X - Y^{(n+1)\varepsilon}$.

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] \\
C_{odd} \arrow[d, "\Delta"] \arrow[rr, equal] & & C_{odd} \arrow[d, "\Delta'"] \\
C_{even} \arrow[d] \arrow[rr, equal] & & C_{even} \arrow[d] \\
0 & & 0
\arrow[from=2-1, to=3-3, phantom, "\underset{(n+3)\varepsilon}{\sim}"]
\end{tikzcd}$$

$\because \ \Delta' = d+\Gamma'$ over $X - Y^{(n+1)\varepsilon}$.

$\therefore \ [\Delta] = [\Delta'] \in Wh(X, Y^{n\varepsilon}, p_X, \ (n+3)\varepsilon)$

2.5により, $[\Delta'] = [C_{\Gamma'}] \in Wh(X, Y^{n\varepsilon}, p_X, \ (n+3)\varepsilon)$

$\therefore [C_\Gamma] = [C_{\Gamma'}] \in Wh(X, Y^{n\varepsilon}, p_X, (n+3)\varepsilon)$

$$[\hat{C}_{\hat{\Gamma}'}] = [\hat{C}_{\hat{\Gamma}}] \in Wh(X, (Y^{60\varepsilon})^{n\cdot 81\varepsilon}, p_X, (n+3)\cdot 81\varepsilon)$$

$$\therefore [C_{\Gamma}] = [C'_{\Gamma'}] \in Wh(X, Y^{(81n+60)\varepsilon}, p_X, (81n+243)\varepsilon)$$

最後に trivial complex を $\oplus$ することは C_{Γ} でも trivial complex を $\oplus$ することに対応しているから C_{Γ} の class は不変.

□

Proposition 2.8

$f: C \to D$, $g: D \to E$ ε chain equivalences.

C, D, E : n-dim free ε chain complexes on p_X.

$\Rightarrow \tau(gf) = \tau(g) + \tau(f) \in Wh(X, p_X, n+1, 18\varepsilon)$

$n=0$ $\tau(gf) = \tau(g) + \tau(f) \in Wh(X, p_X, 2\varepsilon)$.

$\because$ $\mathcal{C}(f) \oplus \mathcal{C}(g) \underset{\text{simple}}{\xrightarrow[\simeq]{}} \mathcal{C}(gf) \oplus \underset{T}{\boxed{\text{trivial}}})$

$\{T, d_T\}$: trivial $T_r = D_r \oplus D_{r-1} \longrightarrow D_{r-1} \oplus D_{r-2} = T_{r-1}$

$$d_T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

ε

① $\begin{pmatrix} 1 & & & (-)^r k \\ & 1 & & f^{-1} \\ & & 1 & \\ & & & 1 \end{pmatrix} : (\mathcal{C}(f) \oplus \mathcal{C}(g))_r = D_r \oplus C_{r-1} \oplus E_r \oplus D_r \to \mathcal{C}(H)_r$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{D_r \oplus C_{r-1} \oplus E_r \oplus D_{r-1}}$

(但し. H, k, f^{-1} は 次ページ).

② $\begin{pmatrix} 1 & & & \\ & 1 & & \\ g & & 1 & \\ & f & & 1 \end{pmatrix} : D_r \oplus C_{r-1} \oplus E_r \oplus D_{r-1} \to D_r \oplus C_{r-1} \oplus E_r \oplus D_{r-1}$

③ $\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ (-)^r d & & & 1 \end{pmatrix}$: 〃 $\to$ 〃

④ $\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & (-)^r \end{pmatrix}$ 〃 $\to$ 〃

の合成で定める。

$$\underbrace{\overset{2\varepsilon}{(\text{boundary})}\overset{0}{\textcircled{4}}}_{\substack{\wr\,2\varepsilon \\ \textcircled{4}\ \partial.}}\ \overset{\varepsilon}{\textcircled{3}}\ \overset{\varepsilon}{\textcircled{2}}\ \overset{\varepsilon}{\textcircled{1}}\ \underset{5\varepsilon}{\sim}\ \overset{0}{\textcircled{4}}\ \underbrace{\partial\ \textcircled{3}}_{\wr\,2\varepsilon}\ \overset{\varepsilon}{\textcircled{2}}\ \overset{\varepsilon}{\textcircled{1}}\ \underset{4\varepsilon}{\sim}\ \overset{0}{\textcircled{4}}\ \overset{\varepsilon}{\textcircled{3}}\ \underbrace{\partial\ \overset{\varepsilon}{\textcircled{2}}}_{\wr\,2\varepsilon}\ \overset{\varepsilon}{\textcircled{1}}$$

$$\underset{4\varepsilon}{\sim}\ \overset{0}{\textcircled{4}}\ \overset{\varepsilon}{\textcircled{3}}\ \overset{\varepsilon}{\textcircled{2}}\ \underbrace{\partial\ \textcircled{1}}_{\substack{\wr\,2\varepsilon \\ \textcircled{1}\ \partial}}\ \underset{4\varepsilon.}{\sim}\ \textcircled{4}\ \textcircled{3}\ \textcircled{2}\ \textcircled{1}\ \partial$$

∴ <u>5ε-simple</u> isomorphism .

$$\begin{tikzcd} f : C \arrow[r] & D \arrow[l, dashed, bend left, "f^{-1}"] \end{tikzcd} , \qquad \begin{tikzcd} g : D \arrow[r] & E \arrow[l, dashed, bend left, "g^{-1}"] \end{tikzcd}$$

$$dh + hd \sim 1 - f^{-1}f \qquad dh' + h'd \sim 1 - g^{-1}g$$
$$dk + kd \sim 1 - ff^{-1} \qquad dk' + k'd \sim 1 - gg^{-1} .$$

$$H = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} : \Omega\mathcal{C}(g)_r = E_{r+1} \oplus D_r \longrightarrow \mathcal{C}(f)_r = D_r \oplus C_{r-1}$$

$$d_{\mathcal{C}(H)} = \left(\begin{array}{cc|cc} d_D & (-)^{r-1}f & & (-)^r \\ & d_C & & \\ \hline & & d_E & (-)^{r-1}g \\ & & & d_D \end{array}\right)$$

$$: D_r \oplus C_{r-1} \oplus E_r \oplus D_{r-1} \longrightarrow D_{r-1} \oplus C_{r-2} \oplus E_{r-1} \oplus D_{r-2} .$$

$$(d_{\mathcal{C}(f)} \oplus d_{\mathcal{C}(g)})_r$$

$$\downarrow = \begin{pmatrix} 1 & & & (-)^r k \\ & 1 & & f^{-1} \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} d & (-)^{r-1}f & & \\ & d & & \\ & & d & (-)^{r-1}g \\ & & & d \end{pmatrix} \qquad \mathcal{C}(f) \oplus \mathcal{C}(g) \longrightarrow \mathcal{C}(H) .$$

$$\left(\begin{array}{c|ccc} 1 & & & (-)^{r-1} k \\ & 1 & & f^{-1} \\ & & 1 & 0 \\ & & & 1 \end{array}\right)$$

$$= \begin{pmatrix} d & (-)^{r-1}f & & (-)^r kd \\ & d & & f^{-1}d \\ & & d & (-)^{r-1}g \\ & & & d \end{pmatrix} \qquad = (-)^r\{-dk + (1 - ff^{-1})\}$$

$$\downarrow = \begin{pmatrix} d_D & (-)^{r-1}f & & (-)^r \\ & d_C & & \\ & & d_E & (-)^{r-1}g \\ & & & d_D \end{pmatrix} \begin{pmatrix} 1 & & & (-)^{r+1}k \\ & 1 & & f^{-1} \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} d_D & (-)^{r-1}f & 0 & (-)^{r+1}dk + (-)^{r-1}ff^{-1} + (-)^r \\ 0 & d_C & 0 & d_C f^{-1} \\ 0 & 0 & d_E & (-)^{r-1}g \\ 0 & 0 & 0 & d_D \end{pmatrix} \underset{2\varepsilon}{\sim} \downarrow$$

$$\mathcal{C}(H)_r = D_r \oplus C_{r-1} \oplus E_r \oplus D_{r-1} \longrightarrow \left(\mathcal{C}(gf) \oplus \mathcal{C}(-1_D)\right)_r$$

$$\parallel$$

$E_r \oplus C_{r-1} \oplus D_r \oplus D_{r-1}$ を

$$\begin{pmatrix} g & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & f & 0 & 1 \end{pmatrix}$$ で定める.

$$↴ = \begin{pmatrix} d_E & (-)^{r-1}gf & & \\ & d_C & & \\ & & d_D & (-)^r \\ & & & d_D \end{pmatrix}\begin{pmatrix} g & & 1 & \\ & 1 & & \\ 1 & & & \\ & f & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} d_E g & (-)^{r-1}gf & d_E & 0 \\ 0 & d_C & 0 & 0 \\ d_D & (-)^r f & 0 & (-)^r \\ 0 & d_D f & 0 & d_D \end{pmatrix}$$

$$↳ = \begin{pmatrix} g & & 1 & \\ & 1 & & \\ 1 & & & \\ & f & & 1 \end{pmatrix}\begin{pmatrix} d_D & (-)^{r-1}f & & (-)^r \\ & d_C & & \\ & & d_E & (-)^{r-1}g \\ & & & d_D \end{pmatrix}$$

$$= \begin{pmatrix} g d_D & (-)^{r-1}gf & d_E & \cancel{(-)^r g} + (-)^{r-1}g \\ 0 & d_C & 0 & 0 \\ d_D & (-)^{r-1}f & 0 & (-)^r \\ 0 & f d_C & 0 & d_D \end{pmatrix}$$

$$\therefore\ ↴ \underset{2\varepsilon}{\sim} ↳$$

$D_r \oplus C_{r-1} \oplus E_r \oplus D_{r-1} \to D_r \oplus C_{r-1} \oplus E_r \oplus D_{r-1}$ では $\left(\begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ \hline g & & 1 & \\ & f & & 1 \end{array}\right)$ とかける。

$$\begin{tikzcd}
\mathcal{C}(-1_D)_r = D_r \oplus D_{r-1} \arrow[r, "\begin{pmatrix} 1 & 0 \\ (-)^r d & 1 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} d & (-)^r \\ 0 & d \end{pmatrix}"'] & D_r \oplus D_{r-1} \arrow[d, "\begin{pmatrix} 0 & (-)^r \\ 0 & 0 \end{pmatrix}"] \\
D_{r-1} \oplus D_{r-2} \arrow[r, "\begin{pmatrix} 1 & 0 \\ (-)^{r-1} d & 1 \end{pmatrix}"'] & D_{r-1} \oplus D_{r-2}
\end{tikzcd}$$

⤵ $= \begin{pmatrix} 0 & (-)^r \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (-)^r d & 1 \end{pmatrix} = \begin{pmatrix} d & (-)^r \\ 0 & 0 \end{pmatrix}$

↳ $= \begin{pmatrix} 1 & 0 \\ (-)^{r-1} d & 1 \end{pmatrix} \begin{pmatrix} d & (-)^r \\ 0 & d \end{pmatrix} = \begin{pmatrix} d & (-)^r \\ (-)^{r-1} d^2 & -d+d \end{pmatrix} \sim_{2\varepsilon}$ ⤵

$$\begin{tikzcd}
D_r \oplus D_{r-1} \arrow[r, "\begin{pmatrix} 1 & 0 \\ 0 & (-)^r \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 0 & (-)^r \\ 0 & 0 \end{pmatrix}"] \arrow[dr, phantom, "\circlearrowright"] & D_r \oplus D_{r-1} \arrow[d, "\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}"] \\
D_r \oplus D_{r-1} \arrow[r, "\begin{pmatrix} 1 & 0 \\ 0 & (-)^{r-1} \end{pmatrix}"'] & D_r \oplus D_{r-1}
\end{tikzcd}$$

⤵ = ↳

§5

$$\begin{tikzcd}
(C,1) \simeq_{\varepsilon}^{\exists} (D,\delta) \ (\text{on } P_W) \arrow[d, Rightarrow, "\text{easy} \quad 3.1\text{ 前半}"] \\
C \text{ is } \varepsilon \text{ dominated by } D \ (\text{on } P_W) \arrow[u, bend right, "3.1\text{ 後半.}"'] \arrow[d, Rightarrow, "\text{easy} \quad 3.2\text{ 前半}"] \\
C \text{ is } \varepsilon \text{ contractible over } X - W^{\varepsilon}. \arrow[u, bend right, dashed, "3.\cancel{2}2\text{ 後半}"']
\end{tikzcd}$$

Proposition 3.1

① C : a free chain complex on $\mathcal{P}_X$

(D, r) : a projective chain complex on $\mathcal{P}_W$, $W \subset X$.

$$\begin{tikzcd}(C,1) \arrow[r, "f"] & (D,r) \arrow[l, bend left, "g"]\end{tikzcd} \quad ; \ \delta \text{ chain eq.}$$

$$h : gf \underset{\delta}{\simeq} 1 \ , \quad k : fg \underset{\delta}{\simeq} 1$$

$\Rightarrow (D, f, g, h)$ is a δ domination of C . $\square$

$gf \underset{\delta}{\simeq} 1$ (i.e. $dh + hd \sim_{2\delta} 1 - gf$)

② (D, f, g, h) : δ domination of C

D : an n-dim free δ chain complex on $\mathcal{P}_W$.

C' を次のように定める.

$$C'_i = D_0 \oplus D_1 \oplus \cdots \oplus D_i$$

$$d'_{2i} = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & \cdots & 2i-1 & 2i \\ \hline 0 & fg & -d & 0 & 0 & \cdots & 0 & 0 \\ 1 & -fhg & 1-fg & d & 0 & \ddots & 0 & 0 \\ 2 & fh^2g & fhg & fg & -d & \ddots & \vdots & \vdots \\ 3 & -fh^3g & -fh^2g & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & & & -d & 0 \\ 2i-1 & -fh^{2i-1}g & -fh^{2i-2}g & & & & 1-fg & d \end{array}$$

$: C'_{2i} = D_0 \oplus D_1 \oplus \cdots \oplus D_{2i} \longrightarrow C'_{2i-1} = D_0 \oplus D_1 \oplus \cdots \oplus D_{2i-1}$,

$$d'_{2i+1} = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & & 2i & 2i+1 \\ \hline 0 & 1-fg & d & 0 & 0 & & 0 & 0 \\ 1 & fhg & fg & -d & 0 & & 0 & 0 \\ 2 & -fh^2g & -fhg & 1-fg & d & \ddots & \vdots & \vdots \\ \vdots & fh^3g & \vdots & & \ddots & & & \\ \vdots & \vdots & & & & & -d & 0 \\ 2i & -fh^{2i}g & -fh^{2i-1}g & -fh^{2i-2} & & & 1-fg & d \end{array}$$

$: C'_{2i+1} = D_0 \oplus \cdots \oplus D_{2i+1} \longrightarrow C'_{2i} = D_0 \oplus D_1 \oplus \cdots \oplus D_{2i}$

役に立つ公式

(A) $h^{2k} gf - h^{2k-1} gf h + h^{2k-2} gf h^2 - \cdots + gf h^{2k}$

$$\underset{(2k+2)\delta}{\sim} h^{2k} - dh^{2k+1} - h^{2k+1} d$$

(B) $h^{2k+1} gf - h^{2k} gf h + h^{2k-1} gf h^2 - \cdots - gf h^{2k+1}$

$$\underset{(2k+3)\delta}{\sim} dh^{2k+2} - h^{2k+2} d .$$

証明　仮定より、$gf \underset{2\delta}{\sim} 1 - dh - hd$ である。 ：(A)の $k=0$. ---(*)

$$hgf - gfh \underset{3\delta}{\sim} h(1-dh-hd) - (1-dh-hd)h$$

$$= h - hdh - h^2 d - h + dh^2 + hdh$$

$$\underset{3\delta}{\sim} dh^2 - h^2 d$$ ：(B)の $k=0$.

他も同様. (*)を代入して整理すればよい。

又は．前の方から2つずつ ペアにして.

(A): $h^{2k-1}(hgf - gfh) + h^{2k-3}(hgf - gfh)h^2 + \cdots \quad + \overset{アマリ}{gfh^{2k}}$

$$\underset{(2k+2)\delta}{\sim} h^{2k-1}(dh^2 - h^2 d) + h^{2k-3}(dh^2 - h^2 d)h^2 + \cdots$$

$$\cdots + h(dh^2 - h^2 d)h^{2k-2} + (1 - dh - hd)h^{2k}$$

$$\underset{(2k+2)\delta}{\sim} -h^{2k+1} d + hdh^{2k} + h^{2k} - dh^{2k+1} - hdh^{2k}$$

$$\underset{(2k+2)\delta}{\sim} h^{2k} - dh^{2k+1} - h^{2k+1} d .$$

(B): $h^{2k}(hgf - gfh) + \cdots \quad + (hgf - gfh)h^{2k}$

$$\underset{(2k+3)\delta}{\sim} h^{2k}(dh^2 - h^2 d) + \cdots + (dh^2 - h^2 d)h^{2k}$$

$$\underset{(2k+3)\delta}{\sim} dh^{2k+2} - h^{2k+2} d .$$

Claim $d'^2 \sim_{(n+4)s} 0$.

㊟ 行列は 行、列とも 0から番号をつける.

(証) $d'_{2i}\, d'_{2i+1}$ の 計算.

d'_{2i} の

$$\begin{array}{lcccccccccc} & 0 & 1 & 2 & \cdots & 2r-1 & 2r & 2r+1 & 2r+2 & & 2i \\ 2r\text{行} = (& f\hbar^{2r}g & f\hbar^{2r-1}g & f\hbar^{2r-2}g & \cdots & f\hbar g & fg & -d & 0 & \cdots & 0\ ) \\ 2r+1\text{行} = (& -f\hbar^{2r+1}g & -f\hbar^{2r}g & -f\hbar^{2r-1}g & \cdots & -fh^2g & -fhg & 1-fg & d & 0\cdots & 0\ ) \end{array}$$

d'_{2i+1} の 2S列は　　　　2S+1列は.

$$\begin{array}{r} 0 \\ 1 \\ \vdots \\ 2s-2 \\ 2s-1 \\ 2s \\ 2s+1 \\ 2s+2 \\ 2s+3 \\ \vdots \\ 2i \end{array} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -d \\ 1-fg \\ fhg \\ -fh^2g \\ fh^3g \\ \vdots \\ -fh^{2i-2s}g \end{pmatrix} \qquad \begin{array}{r} 0 \\ 1 \\ \vdots \\ 2s-2 \\ 2s-1 \\ 2s \\ 2s+1 \\ 2s+2 \\ 2s+3 \\ \vdots \\ 2i \end{array} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ d \\ fg \\ -fhg \\ fh^2g \\ \vdots \\ -fh^{2i-2s-1}g \end{pmatrix}$$

積の $(2r, 2s)$ 成分

場合1. $2s-2 \geqq 2r+1$ のとき $\rightsquigarrow 0$.

つまり. $2s \geqq 2r+3$

$\therefore\ s \geqq r+2$.

場合2. $s = r+1$ のとき.

$(2s-1 = 2r+1)$ 　　　$(-d)(-d) \sim_{2s} 0$.

場合3. $s = r$ のとき. $(2r = 2s)$

$$(f\hbar g)(-d) + fg(1-fg) + (-d)(fhg)$$

$$\sim_{4s} -f\hbar dg + fg - f\boxed{gf}g - fdhg$$

$$= f(1 - gf - (d\hbar + \hbar d))g$$

$$\sim_{4s} 0.$$

場合4. $s<r$ のとき.

ふたつの"ベクトル"をならべてかくと‥

$$\begin{array}{ccccccccc} & 0 & 1 & \cdots & 2s-2 & 2s-1 & 2s & 2s+1 & 2s+2 \\ (& fh^{2r}g & fh^{2r-1}g & \cdots & fh^{2r-2s+2}g & fh^{2r-2s+1}g & fh^{2r-2s}g & fh^{2r-2s-1}g & fh^{2r-2s-2}g \\ (& 0 & 0 & \cdots & 0 & -d & 1-fg & fhg & -fh^2g \end{array}$$

$$\begin{array}{ccccccc} & 2r-1 & 2r & 2r+1 & 2r+2 & & 2i \\ \cdots & fhg & fg & -d & 0 & \cdots & 0\ ) \\ \cdots & fh^{2r-1-2s}g & -fh^{2r-2s}g & fh^{2r-2s+1}g & \cdots & & -fh^{2i-2s}g) \end{array}$$

$$\therefore\ -fh^{2r-2s+1}\underline{\underline{g\cdot d}} + fh^{2r-2s}g(1-fg) + fh^{2r-2s-1}g\,fhg - fh^{2r-2s-2}g\,fh^2g + \cdots$$
$$\cdots + fhg\,fh^{2r-1-2s}g - fg\,fh^{2r-2s}g - \underline{\underline{df}}h^{2r-2s+1}g\ .$$

$$\underset{(2r-2s+4)\delta}{\sim}\ f\Big[-\underline{h^{2r-2s+1}d} + \underline{h^{2r-2s}} - h^{2r-2s}gf + h^{2r-2s-1}gfh - \cdots$$
$$\cdots + hgfh^{2r-1-2s} - gfh^{2r-2s} - \underline{dh^{2r-2s+1}}\Big]g$$

$$\underset{(2r-2s+4)\delta}{\sim}\ f\Big[h^{2r-2s} \underbrace{-dh^{2r-2s+1} - h^{2r-2s+1}d - \{h^{2r-2s}gf - \cdots\}}_{\underset{(2r-2s+2)\delta}{\wr}\ 0}\Big]g$$

$$\underset{(2r-2s+4)\delta}{\sim}\ 0$$

$(2r, 2s+1)$-成分.

場合1. $2r+1 \leqq 2s-1$ (つまり $s \geqq r+1$) $\Rightarrow 0$.

場合2. $s=r$ のとき.

$$fgd - dfg \underset{\substack{3\varepsilon \\ (2\varepsilon でもよい)}}{\sim} dfg - dfg \underset{\substack{3\varepsilon \\ (2\varepsilon)}}{\sim} 0\ .$$

場合3. $s<r$ のとき.

$$\begin{array}{cccccccccc} & & 2s & 2s+1 & 2s+2 & \cdots & 2r-1 & 2r & 2r+1 & \\ (& fh^{2r}g\ \cdots & fh^{2r-2s}g & fh^{2r-2s-1}g & fh^{2r-2s-2}g & & fhg & fg & -d & 0\cdots 0) \\ (& 0\ \cdots & d & fg & -fhg & & -fh^{2r-2s-1}g & fh^{2r-2s}g & \cdots &) \end{array}$$

$$\therefore\ fh^{2r-2s}\underline{gd} + \overbrace{fh^{2r-2s-1}g\,fg - fh^{2r-2s-2}g\,fhg + \cdots - fg\,fh^{2r-2s-1}g} - \underline{df}h^{2r-2s}g$$

$$\underset{(2r-2s+3)\delta}{\sim} f\Big[\underbrace{h^{2r-2s}d - dh^{2r-2s} + \{h^{2r-2s-1}gf - h^{2r-2s-2}gfh + \cdots - gfh^{2r-2s-1}\}}_{\sim_{(2r-2s+1)\delta}\ 0}\Big]g$$

$$\underset{(2r-2s+3)\delta}{\sim} 0$$

(2r+1 , 2s) 成分.

場合1. $2s-2 \geqq 2r+2$ i.e. $s \geqq r+2$ のとき 0.

場合2. $s=r+1$ のとき $(1-fg)(-d)+d(1-fg) \underset{\substack{3\varepsilon \\ (2\varepsilon)}}{\sim} 0$

場合3. $s \leqq r$ のとき.

$$\begin{array}{lllllllll}
 & 2s-1 & 2s & 2s+1 & & 2r & 2r+1 & 2r+2 & \\
(-fh^{2r+1}g \cdots & -fh^{2r-2s+2}g & -fh^{2r-2s+1}g & -fh^{2r-2s}g & \cdots & -fhg & 1-fg & d & 0 \cdots) \\
(\ \ 0 \quad \cdots & -d & 1-fg & fhg & \cdots & -fh^{2r-2s}g & fh^{2r-2s+1}g & -fh^{2r-2s+2}g & \cdots)
\end{array}$$

$$fh^{2r-2s+2}\underline{\underline{gd}} - fh^{2r-2s+1}g + fh^{2r-2s+1}gfg - fh^{2r-2s}gfhg + \cdots + fhgfh^{2r-2s}g$$
$$+ fh^{2r-2s+1}g - fgfh^{2r-2s+1}g - \underline{\underline{df}}\,h^{2r-2s+2}g$$

$$\underset{(2r-2s+5)\delta}{\sim} f\Big[h^{2r-2s+2}d - dh^{2r-2s+2} + \{h^{2r-2s+1}gf - h^{2r-2s}gfh + \cdots - gfh^{2r-2s+1}\}\Big]g$$

$$\underset{(2r-2s+5)\delta}{\sim} 0.$$

(2r+1, 2s+1) 成分

場合1. $2s-1 \geqq 2r+2$ ($s \geqq r+2$) 0

場合2. $s=r+1$ ($2s=2r+2$) $dd \underset{2\varepsilon}{\sim} 0$.

場合3. $s=r$ のとき.

$$-fhg\cdot d + (1-fg)fg + d(-fhg)$$

$$\underset{4\delta}{\sim} f[-hd+1-gf-dh]g \underset{4\delta}{\sim} 0.$$

場合4. $s<r$ のとき.

$$\begin{array}{lllllllll}
 & 2s & 2s+1 & 2s+2 & & 2r & 2r+1 & 2r+2 & \\
(-fh^{2r+1}g \cdots & -fh^{2r-2s+1}g & -fh^{2r-2s}g & -fh^{2r-2s-1}g & \cdots & -fhg & 1-fg & d & 0 \cdots) \\
(\ \ 0 & d & fg & -fhg & \cdots & -fh^{2r-2s-1}g & fh^{2r-2s}g & -fh^{2r-2s+1}g & \cdots)
\end{array}$$

$$\therefore\ -fh^{2r-2s+1}\underline{gd} + fh^{2r-2s}gfg + fh^{2r-2s-1}gfhg - \cdots + fhgfh^{2r-2s-1}g$$

$$+ fh^{2r-2s}g \boxed{- fgfh^{2r-2s}g} - \underline{\underline{d}}fh^{2r-2s+1}g$$

$$\underset{(2r-2s+4)\delta}{\sim}\ f\left[-h^{2r-2s+1}d + h^{2r-2s} - dh^{2r-2s+1} - \left\{h^{2r-2s}gf - h^{2r-2s-1}gfh + \cdots\right\}\right]g$$

$$\underset{(2r-2s+4)\delta}{\sim}\ 0 .$$

以上まとめると

$\sim$ の最大は

$\underbrace{f}_{\varepsilon}, \underbrace{g}_{\varepsilon}, \underbrace{f}_{\varepsilon}, \underbrace{g}_{\varepsilon}$, h(何回も) の積の size でおさえられる!!

(例) $\underline{fgfh^{2r-2s}g}$

hの巾は最大で $n\delta$　　$h^{n+1}=0$!!

$$\begin{tikzcd} C_0 \arrow[r, "h"] & C_1 \arrow[r, "h"] & \cdots \arrow[r, "h"] & C_n \end{tikzcd}$$

$\therefore$ $(n+4)\delta$ が使える。

実は $i=\mathrm{odd}$ も同様.　$\therefore\ d'^2 \underset{(n+4)\delta}{\sim} 0$.

C と C' は chain equivalent である. $$\begin{tikzcd} C \arrow[r, shift left, "f'"] & C' \arrow[l, shift left, "g'"] \end{tikzcd}$$

$$f' = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f \end{pmatrix} : C_i \longrightarrow C'_i = D_0 \oplus \cdots \oplus D_i \quad \text{: radius } \delta.$$

$$\text{↴} = \begin{pmatrix} * & * & & & 0 \\ & * & * & & \vdots \\ & & * & \ddots & \\ & & & * & 0 \\ & & & * & d \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ \\ 0 \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ df \end{pmatrix}$$

$$\text{↳} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f \end{pmatrix} d = \begin{pmatrix} 0 \\ \vdots \\ \\ fd \end{pmatrix} \qquad \therefore \ \text{↴} \sim_\delta \text{↳}.$$

$\therefore$ f' は δ chain map.

$$g' = (h^i g \ \ h^{i-1} g \ \cdots \ hg \ \ g) : C'_i = D_0 \oplus \cdots \oplus D_i \to C_i$$

$$\begin{tikzcd} C_j \arrow[d] & C'_j \arrow[l] \arrow[d] \\ C_{j-1} & C'_{j-1} \arrow[l] \end{tikzcd}$$

$\vdots$ radius $(n+1)\delta$

$$\text{↓} = d_j (h^j g \ \ h^{j-1} g \ \cdots \ hg \ \ g) = (d_j h^j g \ \ d_j h^{j-1} g \ \cdots \ d_j hg \ \ g)$$

$$\text{↵} = (h^{j-1} g \cdots \ hg \ \ g) \begin{pmatrix} \ddots & \\ & d \end{pmatrix}$$

<u>j : odd のとき.</u>

$$(h^{j-1}g \ \ h^{j-2}g \ \cdots \ hg \ \ g) \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \cdots & & j \end{array} \\ \begin{pmatrix} 1-fg & d & 0 & & & \\ fhg & fg & -d & \ddots & & \\ -fh^2g & -fhg & 1-fg & \ddots & & \\ \vdots & \vdots & & & -d & \\ -fh^{j-1}g & -fh^{j-2}g & \cdots & & & d \end{pmatrix} \end{array}$$

$2s$ 成分

$$= h^{\overset{j-2s}{\boxed{j-1-(2s-1)}}} g(-d) + h^{j-2s-1} g(1-fg) + h^{j-2s-2} g fhg - h^{j-2s-3} g fh^2 g + \cdots - g(fh^{j-1-2s} g)$$

$$\underset{(j-2s+2)\delta}{\sim} \left[-h^{j-2s} d + h^{j-2s-1} - \{ h^{j-2s-1} gf - h^{j-2s-2} gfh + \cdots + gfh^{j-2s-1} \} \right] g$$

$$\underset{(j-2s+2)\delta}{\sim} \quad d h^{j-2s} g$$

最大は $(n+2)\delta$.

$(2s+1)$成分

$$= h^{j-2s-1} g d + h^{j-2s-2} g(fg) - h^{j-2s-3} g(fhg) + \cdots - gfh^{j-2s-2} g$$

$$\underset{(j-2s+1)\delta}{\sim} \left[h^{j-2s-1} d + \left\{ \underline{h^{j-2s-2} gf - h^{j-2s-3} gfh + \cdots - gfh^{j-2s-2}} \right\} \right] g$$

$$\overset{2(j-2s)\delta}{d h^{j-2s-1} - h^{j-2s-1} d}.$$

$$\underset{(j-2s+1)\delta}{\sim} d h^{j-2s-1} g$$

$\therefore$ j : odd のときは $\sim_{(n+2)\delta}$.

j : even のとき

$$2s+1 \text{ 成分} = (h^{j-1}g \quad h^{j-2}g \cdots hg \quad g) \left(\begin{array}{c|c} 0 & 0 \\ \vdots & \vdots \\ 0 & 2s-1 \\ -d & 2s \\ 1-fg & 2s+1 \\ fhg & 2s+2 \\ \vdots & \vdots \\ -fh^{j-(2s+1)-1}g & j-1 \end{array}\right).$$

$$= -h^{j-2s-1} g d + h^{j-2s-2} g(1-fg) + h^{j-2s-3} gfhg - \cdots - gfh^{j-2s-2} g$$

$$\underset{(j-2s+1)\delta}{\sim} \left[-h^{j-2s-1} d + h^{j-2s-2} - \left\{ h^{j-2s-2} gf - \cdots + gfh^{j-2s-2} \right\} \right] g$$

$$\underset{(j-2s+1)\delta}{\sim} d h^{j-2s-1} g$$

$$2s \text{ 成分} = (h^{j-1}g \quad h^{j-2}g \cdots hg \quad g) \left(\begin{array}{c|c} 0 & 0 \\ \vdots & \vdots \\ 0 & \\ d & 2s-1 \\ fg & 2s \\ -fhg & \vdots \\ fh^2 g & \\ \vdots & \\ -fh^{j-2s-3} g & j-1 \end{array}\right)$$

$j-1-(2s\cdots)$

$$= h^{j-2s-2} g d + h^{j-2s-3} gfg - h^{j-2s-4} gfhg + \cdots \cdots - gfh^{j-2s-3} g$$

$$\underset{(j-2s)\delta}{\sim} \left[h^{j-2s-2} d + \left\{ \underline{h^{j-2s-3} gf - h^{j-2s-4} gfh + \cdots - gfh^{j-2s-3}} \right\} \right] g$$

$$\hookrightarrow \underset{(j-2s-1)\delta}{\sim} \left\{ a h^{j-2s-2} - h^{j-2s-2} d \right\}$$

$\sim_{(j-2s)\delta} \quad d k^{j-2s-2}$.

$\therefore$ j: even のときは $\sim_{(n+2)\delta}$.

$\therefore$ g' は $(n+2)\delta$ chain map

$\circlearrowleft$

$$g'f' = (h^i g \quad h^{i-1}g \quad \cdots \quad hg \quad g)\begin{pmatrix} 0 \\ \vdots \\ 0 \\ f \end{pmatrix} = gf \simeq_{\delta} 1 \quad : k \quad \because \text{domination}$$

$\circlearrowright$

$$f'g' = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f \end{pmatrix}(h^i g \quad h^{i-1}g \quad \cdots \quad hg \quad g) : C'_i = D_0 \oplus \cdots \oplus D_i \to D_0 \oplus \cdots \oplus D_i = C'_i$$

$$= \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 0 & 0 \\ fh^i g & fh^{i-1}g & \cdots & fhg & fg \end{pmatrix}$$

$f'g' \simeq 1$ (?)

$$k' : C'_i = D_0 \oplus \cdots \oplus D_i \to D_0 \oplus \cdots \oplus D_i \oplus D_{i+1} = C'_i$$

$$k' = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & 0 \end{pmatrix}$$ と定める。

$$d'k' + k'd' = \begin{pmatrix} 1-fg & d & & \\ fhg & fg & \ddots & \\ & & & \\ -fh^i g & -fh^{i-1}g & \cdots & d \end{pmatrix}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & 0 \end{pmatrix} + \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & 0 \end{pmatrix}\begin{pmatrix} fg & -d & & \\ -fhg & & & \\ \vdots & & & \\ -fh^{i-1}g & & & d \end{pmatrix}$$

$(i = \text{even})$

$$= \begin{pmatrix} 1-fg & d & & \\ fhg & & \ddots & \\ & & & -d \\ -fh^i g & & & 1-fg \end{pmatrix} + \begin{pmatrix} fg & -d & & \\ -fhg & 1-fg & & \\ \vdots & & & \\ -fh^{i-1}g & & & d \\ 0 & & & 0 \end{pmatrix} \sim_{(n+2)\delta} 1 - f'g'$$.

E を C' の n-skeleton とする $\cdots \to 0 \to C'_n \xrightarrow{d'} C'_{n-1} \to \cdots \to C'_1 \xrightarrow{d'} C'_0$.

$g = \{g_i : E_i \to E_i\}$ を

$$g_i = 1 : E_i = C'_i \longrightarrow E_i = C'_i \quad (0 \le i \le n-1)$$

$$g_n = \begin{cases} p & : E_n = C'_n \longrightarrow E_n = C'_n \quad \text{if } n : \text{even} \\ 1-p & : \quad 〃 \quad \text{odd} . \end{cases}$$

但し $p = \begin{pmatrix} fg & -d & 0 & \cdots \\ -fhg & 1-fg & d & \\ fh^2g & fhg & fg & \\ \vdots & \vdots & \vdots & \end{pmatrix}$

$$: C'_n = D_0 \oplus \cdots \oplus D_n \longrightarrow C'_n = D_0 \oplus \cdots \oplus D_n .$$

つまり $g_n = 1 - d'_{n+1} : C'_{n+1} = C'_n \longrightarrow C'_n$.

$d'_i = p$ if i : even, $i > n$

$d'_i = 1-p$ if i : odd , $i > n$.

$$\therefore p(1-p) \underset{(n+4)\delta}{\sim} 0$$

$$\therefore p - p^2 \underset{(n+4)\delta}{\sim} 0$$

$$\therefore p \underset{(n+4)\delta}{\sim} p^2 .$$

$$d'_n g_n \underset{(n+4)\delta}{\sim} d'_n$$

$$\because 0 \underset{(n+4)\delta}{\sim} d'_n d'_{n+1} = d'_n(1-g_n) = d'_n - d'_n g_n .$$

$\therefore (E, g)$ は n-dim $(n+4)\delta$ proj. chain complex .

$$(C', 1) \underset{\overleftarrow{J}}{\overset{\overrightarrow{I}}{\simeq}} (E, g) . \text{ を } I = \begin{cases} 1 & : C'_i \longrightarrow E_i = C'_i & \text{if } 0 \le i \le n-1 \\ g_n & : C'_i \longrightarrow E_i = C'_i & \text{if } i = n \\ 0 & : C'_i \longrightarrow E_i = 0 & \text{if } i > n . \end{cases}$$

$$J = \begin{cases} 1 & : E_i \longrightarrow C'_i & \text{if } 0 \le i \le n-1 \\ g_n & : E_i \longrightarrow C'_i & \text{if } i = n \\ 0 & : E_i \longrightarrow C'_i & \text{if } i > n . \end{cases}$$

共に $(n+4)\delta$ morphism である。, さらに $(n+4)\delta$ chain map.

$IJ = 1,\ g_n,\ 0$

$\therefore IJ \sim_{(n+2)\delta} g$. $(=\ )$.

$JI = 1,\ g_n,\ 0$ $\quad\therefore 1 - JI = 0,\ 1-g_n,\ 1$

$$K = \begin{cases} 0 : C'_i \longrightarrow C'_{i+1} & \text{if} \quad 0 \leq i \leq n-1 \\ 1 : C'_i \longrightarrow C'_{i+1} & \text{if} \quad i \geq n . \end{cases}$$

$$\begin{tikzcd}
D_0 \oplus \cdots \oplus D_n \arrow[d, shift right, "d'_{n+1}"'] \\
D_0 \oplus \cdots \oplus D_n \arrow[u, shift right, "1"'] \arrow[d, shift right, "d'_n"'] \\
D_0 \oplus \cdots \oplus D_{n-1} \arrow[u, shift right, "0"'] \arrow[d, shift right, "d'_{n-1}"'] \\
\cdots \arrow[u, shift right, "0"']
\end{tikzcd}$$

- $d'K + Kd' = 0$ if $i \leq n-1$.
- $d'K + Kd' = d'_{n+1}$ if $i = n$.
  $= 1 - g_n$.
- $d'K + Kd' = d'_i + d'_{i+1}$ if $i > n$.
  $= 1$

$\therefore d'K + Kd' \sim_{(n+2)\delta} 1 - JI$.

$\therefore$ I, J is $(n+4)\delta$ chain equivalence

Proposition 3.2. <u>domination</u>

$$\begin{tikzcd}(C, 1) \arrow[r, "f"] & (D, \delta) \arrow[l, bend left, "g"]\end{tikzcd}$$

~~$g : (D, \delta) \to (C, 1)$: ε chain equivalences.~~

$$d\tilde{h} + \tilde{h}d \sim_{2\varepsilon} 1 - gf$$ ~~$d\tilde{k} + \tilde{k}d \sim_{2\varepsilon} \delta - \delta\delta$.~~

f の radius は ε だから. $X - W^{\varepsilon}$ では f は 0 でなければならない.

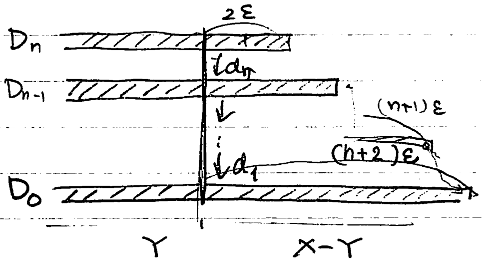
$$\therefore dh + hd \sim_{2\varepsilon} 1 \quad \text{over } X - W^{\varepsilon}.$$

~~Proposition 3.3~~ 後半

C : n-dim free ε chain complex, ε contractible over $X - Y$.

$d\Gamma + \Gamma d \sim_{2\varepsilon} 1$ over $X - Y$. とする.

define: $D_r = C_r(Y^{(n-r+2)\varepsilon})$



$d_r = d_r | Y^{(n-r+2)\varepsilon}$

<u>$d^2 \sim 0$ を言えねばならぬのは</u>

$D_2 \to D_0$ 以上.

∴ これは $p_{Y^{(n+2)\varepsilon}}$ 上と見える。

$$1 - d\Gamma - \Gamma d \sim_{2\varepsilon} 0 \quad \text{over } X - Y$$

$$\text{define } F = \begin{cases} 0 & \text{over } X - Y \\ 1 - d\Gamma - \Gamma d & \text{over } Y \end{cases}, \quad \text{then } \underset{\overset{\shortparallel}{if}}{F} \sim_{2\varepsilon} 1 - d\Gamma - \Gamma d$$

then

$$\begin{tikzcd} & D_r \arrow[dr, "i"] & \\ C_r \arrow[ur, dashed, "f_r"] \arrow[rr, "F_r"] & & C_r \end{tikzcd}$$

$$F_r : C_r \longrightarrow C_r$$

$$i(d_D f - f d_C) = dF - Fd \sim_{3\varepsilon} d(1 - d\Gamma - \Gamma d) - (1 - d\Gamma - \Gamma d)d$$

$$= d - dd\Gamma - d\Gamma d - d + d\Gamma d + \Gamma dd \sim_{3\varepsilon} -dd\Gamma + \Gamma dd \sim_{3\varepsilon} 0$$

$\partial : Wh(X, Y, p_X, n, \varepsilon) \longrightarrow \tilde{K}_0(W, p_W, n, \varepsilon')$ の def

$\overset{\Psi}{[C]}$

C is (strongly) ε contractible over $X-Y$

$\downarrow$ 3.2 より

C is 3ε dominated by an n-dim f.g. free ε chain complex D on $P_{Y^{(n+2)\varepsilon}}$.

$\downarrow$ 3.1 より.

$(C,1)$ is $(2n+5)\cdot 3\varepsilon$ chain eq. to an n-dim f.g. $(n+4)\cdot 3\varepsilon$ projective chain complex on $P_{\left(Y^{(n+2)\varepsilon}\right)^{(n+4)3\varepsilon}}$

(E, g)

$\cap$ $Y^{(4n+14)\varepsilon}$

$$(C', 1) \underset{(6n+15)\varepsilon}{\simeq} (E', g') \quad \text{on } P_{Y^{(4n+14)\varepsilon}}$$

$$C \oplus D \oplus T \underset{40\varepsilon, \Sigma}{\cong} C' \oplus D' \oplus T'$$

$$\begin{array}{ccc} (C,1) \oplus (D,1) & \simeq_{40\varepsilon} & (C',1) \oplus (D',1) \\ \wr| & & \wr| \\ (E,g) & & (E',g') \end{array}$$

$$(E,g) \oplus \underbrace{(D,1)}_{} \underset{(12n+70)\varepsilon}{\simeq} (E',g') \oplus \underbrace{(D',1)}_{}$$

free 40ε ch cx on $Y^{20\varepsilon}$

free 40ε ch. cx.

on $Y^{(4n+14)\varepsilon}$

あわせて $Y^{(4n+20)\varepsilon}$

Remark.

Lemma 3.4

(1) $h : f \underset{\delta}{\simeq} f'$ $\quad dh + hd \sim_{2\delta} f' - f$.

$$\begin{pmatrix} 1 & (-)^r h \\ 0 & 1 \end{pmatrix} : \mathcal{C}(f)_r = D_r \oplus C_{r-1} \to \mathcal{C}(f')_r = D'_r \oplus C'_{r-1}$$

radius δ. $\quad$ δ-simple isomaphis

$$\rightarrow\!\downarrow = \begin{pmatrix} d'_r & (-)^{r-1} f' \\ 0 & d'_r \end{pmatrix} \begin{pmatrix} 1 & (-)^r h \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} d' & (-)^r d'h - (-)^r f' \\ 0 & d' \end{pmatrix}$$

$$\downarrow\!\rightarrow = \begin{pmatrix} 1 & (-)^{r-1} h \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & (-)^{r-1} f \\ 0 & d \end{pmatrix} = \begin{pmatrix} d & (-)^{r-1} f + (-)^{r-1} hd \\ 0 & d \end{pmatrix}$$

$\therefore \rightarrow\!\downarrow \sim_{2\delta} \downarrow\!\rightarrow$ $\quad \therefore$ 2δ chain map

$\leftarrow$ も 2δ chain map $\quad \therefore$ 2δ-simple isom.

(2) $T_r = C_r \oplus C_{r-1} \xrightarrow[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}]{d_r} C_{r-1} \oplus C_{r-2} = T_{r-1}$: <u>trivial</u>

$$\mathcal{C}(1_C)_r = C_r \oplus C_{r-1} \xrightarrow{\begin{pmatrix} (-)^r 1 & 0 \\ (-)^{r-1} d_C & 1 \end{pmatrix}} C_r \oplus C_{r-1} = T_r$$

radius δ

$$\rightarrow\!\downarrow = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (-)^r 1 & 0 \\ (-)^{r-1} d_C & 1 \end{pmatrix} = \begin{pmatrix} (-)^{r-1} d_C & 1 \\ 0 & 0 \end{pmatrix}$$

$$\downarrow\!\rightarrow = \begin{pmatrix} (-)^{r-1} 1 & 0 \\ (-)^r d_C & 1 \end{pmatrix} \begin{pmatrix} d_C & (-)^{r-1} 1 \\ 0 & d_C \end{pmatrix} = \begin{pmatrix} (-)^{r-1} d_C & 1 \\ (-)^r d_C^2 & -d_C + d_C \end{pmatrix}$$

$\therefore \rightarrow\!\downarrow \sim_{2\delta} \downarrow\!\rightarrow$ $\quad \therefore$ 2δ chain map

逆も同様に 2δ chain map. $\quad \therefore$

3.3

(1) $Wh(X, p_X, n, \varepsilon) \longrightarrow Wh(X, Y, p_X, n, \varepsilon)$

$$[C] \longmapsto 0.$$

2.1により

$\exists\, 86\varepsilon$-simple iso $f : C \oplus D \oplus T \longrightarrow D' \oplus T'$

n-dim 86ε chain cx's on p_Y 20ε

$i : D \to C \oplus D \oplus T$ $\qquad j : D' \to D' \oplus T'$

strongly ε or ' 2ε. $\qquad q$ proj.

$g = qfi$: 100ε chain equivalence !!

radius は

これも86ε.

$$d(qfi) = q\,df\,i \underset{86\varepsilon}{\sim} qfdi = qfid.$$

☺ $0 \simeq T$ 0-chain equiv.

$$\begin{tikzcd} C\oplus D\oplus T \arrow[r, "86\varepsilon"', "f"] & D'\oplus T' \arrow[d, shift left, "q", "86"'] \\ D \arrow[u, "i", "86"'] \arrow[r, dashed, "g"'] & D' \arrow[u, shift left, "j"] \end{tikzcd}$$

$\therefore\ \mathcal{C}(g)$: 300ε contractible

$$\mathcal{C}(g) \oplus T = \mathcal{C}(jg : D \to D' \oplus T')$$
$$jg = jqfi$$

hとおく.

$jq \simeq 1$

$jq = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} : D' \oplus T' \to D' \oplus T'$

0 chain contraction of T' (see next page)

$$\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} D & 0 \\ 0 & \Gamma \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \Gamma \end{pmatrix}\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}$$

$d_{D'}$

d_T

$$= \begin{pmatrix} 0 & 0 \\ 0 & \underset{0}{d\Gamma} + \underset{0}{\Gamma d} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - jq$$

d_T

$$dh + hd = 1 - jq$$
$$dhfi + hdfi = (1 - jq)fi = fi - jqfi$$
$$\wr\,86\varepsilon$$
$$d(hfi) + (hfi)d$$

$\therefore\ jg$ と fi は 100ε chain htpic.

$\therefore\ \mathcal{C}(jg)$ と $\mathcal{C}(fi)$ は 200ε-simple isomorphic elementary trivial chain complex

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & A \arrow[r, "\overset{d}{\overset{\shortparallel}{1}}"] \arrow[l, bend left, "\Gamma=0"] & A \arrow[r] \arrow[l, bend left, dashed, "\Gamma=1"] & 0 \arrow[r] \arrow[l, bend left, "\Gamma=0"] & \cdots
\end{tikzcd}$$

は <u>0 contractible</u> , $\Gamma = 1$.

$$d\Gamma + \Gamma d = 1$$

$\oplus$ をとれば

T: trivial chain complex も 0 contractible

$^{\exists}\Gamma : 0 \simeq 1$.

$$i : D \to \overbrace{\mathcal{C} \oplus D \oplus T}^{A} \xrightarrow[\underset{86\varepsilon,\Sigma}{\cong}]{f} \overbrace{D' \oplus T}^{B}$$

$$(86\varepsilon)$$

f : 186ε-simple iso.

$$\Rightarrow \mathcal{C}(fi) \underset{\substack{\cancel{100}\varepsilon,\Sigma \\ \cancel{200} \\ 172}}{\cong} \mathcal{C}(i)$$

$$\begin{tikzcd}
(\because)\ \mathcal{C}(fi)_r = B_r \oplus D_{r-1} \arrow[d, "\begin{pmatrix} d & (-)^{r-1}fi \\ 0 & d \end{pmatrix}"'] & \mathcal{C}(i)_r = A_r \oplus D_{r-1} \arrow[l, "\begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}"'] \arrow[d, "\begin{pmatrix} d & (-)^{r-1}i \\ 0 & d \end{pmatrix}"] \\
B_{r-1} \oplus D_{r-2} & \mathcal{C}(i)_{r-1} = A_{r-1} \oplus D_{r-2} \arrow[l, "\begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}"]
\end{tikzcd}$$

$$⬐ = \begin{pmatrix} d & (-)^{r-1}fi \\ 0 & d \end{pmatrix}\begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} df & (-)^{r-1}fi \\ 0 & d \end{pmatrix}$$

$$↲ = \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} d & (-)^{r-1}i \\ 0 & d \end{pmatrix} = \begin{pmatrix} fd & (-)^{r-1}fi \\ 0 & d \end{pmatrix}$$

$$86\varepsilon \quad fd \underset{86\varepsilon}{\sim} df$$

$$\therefore ⬐ \underset{86\varepsilon}{\sim} ↲$$

$\therefore$ 86ε chain map.

逆向

$$⬎ = \begin{pmatrix} d & (-)^{r-1}i \\ 0 & d \end{pmatrix}\begin{pmatrix} f^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} df^{-1} & (-)^{r-1}i \\ 0 & 0 \end{pmatrix}$$

$$↳ = \begin{pmatrix} f^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} d & (-)^{r-1}fi \\ 0 & d \end{pmatrix} = \begin{pmatrix} f^{-1}d & (-)^{r-1}f^{-1}fi \\ 0 & d \end{pmatrix}$$

$$⬎ \underset{\substack{2\cdot\cancel{86}\varepsilon \\ 100}}{\sim} ↳ \qquad \text{逆は } \overset{200}{\cancel{172}}\,\varepsilon \text{ chain map.}$$

$\therefore$ $\overset{200}{\cancel{172}}$ε-simple isomorphism.

$$\begin{tikzcd}
Wh(X, p_X, n+1, 900\varepsilon) \arrow[r] \arrow[d, "\tau"] & Wh(X, p_X, n, (90n+340)900\varepsilon) \arrow[d, equal] \\
Wh(X, p_X, 1, (90(n+1)+250)\cdot 900\varepsilon) \arrow[ru, "\iota"'] & 9000(9n+34)\varepsilon
\end{tikzcd}$$

✓

$\varepsilon' \geq (12n+70)\varepsilon$

(2)
$$\begin{tikzcd} Wh(X, Y, p_X, n, \varepsilon) \arrow[r, "\partial"] & \tilde{K}_0(W, p_W, n, \varepsilon') \\ {[C]} \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & 0 \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

$\partial[C] = \underbrace{[E, g]}$,

$$(C, 1) \underset{(6n+15)\varepsilon}{\simeq} (E, g)$$

n-dim f.g. $(3n+12)\varepsilon$ proj. chain cx. on $p_{Y(4n+14)\varepsilon}$

$[E, g] = 0$ in $\tilde{K}_0(W, p_W, n, \varepsilon')$ だから. 1.9 により.

$$\exists (D, 1) \underset{60\varepsilon'}{\simeq} (E, g) \underset{\varepsilon'}{\simeq} (C, 1)$$

on p_W

n-dim. f.g.

free $30\varepsilon'$ chain cx

$$\therefore D \underset{61\varepsilon'}{\overset{f}{\simeq}} C$$

$\mathcal{C}(f)$: $(n+1)$-dim free $61\varepsilon'$ chain complex on p_X

1.4 により. $183\varepsilon'$ contractible.

$\therefore$ strongly $549\varepsilon'$ contractible.

$$\overset{2.5}{[\bar{C}] = [\mathcal{C}(f)] = [C]}$$

$$\begin{tikzcd} {[\mathcal{C}(f)] \in Wh(X, p_X, n+1, 549\varepsilon')} \arrow[r] & Wh(X, W, p_X, n+1, 549\varepsilon') \arrow[d, "\tau"] \\ {\exists[\bar{C}] \in Wh(X, p_X, n, 549\varepsilon')} \arrow[u, mapsto] \arrow[u, shift left=10, "\iota"'] \arrow[r] & Wh(X, W, p_X, n, 549\varepsilon') \arrow[u] \\ & Wh(X, W^{(90(n+1)+100)549\varepsilon'}, p_X, (90(n+1)+250)\cdot 549\varepsilon') \end{tikzcd}$$

2.5 により $[\mathcal{C}(f)] = [C] \in Wh(X, W, p_X, n+1, 549\varepsilon')$

$$\| $$

$$Wh(X, W^{549(90n+190)\varepsilon'}, p_X, 549(90n+340)\varepsilon').$$

(3).

$$\begin{tikzcd}
\tilde{K}_0(W, p_W, n, \varepsilon') \arrow[r, "i_*"] \arrow[d, phantom, sloped, "\ni"] & \tilde{K}_0(X, p_X, n, \varepsilon') \arrow[d, phantom, sloped, "\ni"] \\
{[E, g]} \arrow[r, mapsto] & 0 .
\end{tikzcd}$$

$(E, g) \underset{60\varepsilon'}{\simeq}$ $\exists$ an n-dim f.g. free $\underline{30\varepsilon'}$ chain cx $(C, 1)$ on p_X.

on W.

C is $60\varepsilon'$ contractible over $X - W^{60\varepsilon'}$.

$\therefore$ C is strongly $180\varepsilon'$ contractible over $X - W^{240\varepsilon'}$.

$$\begin{tikzcd}
\therefore [C] \in Wh(X, W^{240\varepsilon'}, p_X, n, 180\varepsilon') . \arrow[d, "\partial"] \\
\tilde{K}_0(W', p_{W'}, n, \delta')
\end{tikzcd}$$

$\square$

- $W' \supset (W^{240\varepsilon'})^{(4n+20)\cdot 180\varepsilon'}$

例えば

$W' = W^{180(4n+22)\varepsilon'}$

- $\delta' \geqslant (12n + 70) \cdot 180\varepsilon'$

§6

exc. の定義.　　　　　　　　　　　　　　　　　　exc.

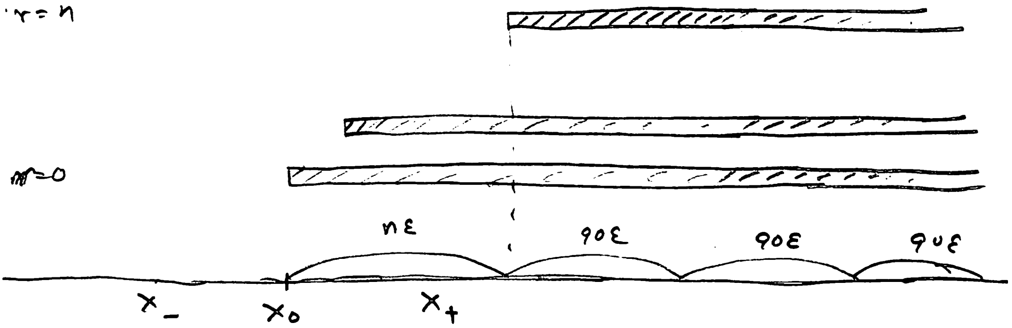
$$Wh(X, X_-, p_X, n, \varepsilon) \longrightarrow Wh(X_+, X_+ \cap X_0 \quad , p_{X_+}, n, \bigcirc)$$

$$\overset{\in}{[C]}$$

C_+: an n-dim f.g. free 90ε chain complex on p_{X_+} s.t.

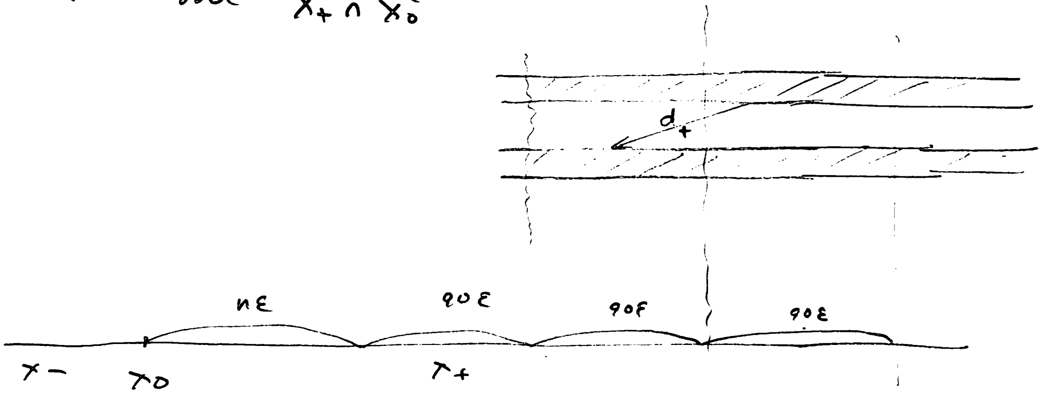
$$\begin{cases} C = C_+ & \text{over } X_+ - X_0^{(n+180)\varepsilon} . \leftarrow \text{module の一致} \\ d_C = d_{C_+} & \text{over } X_+ - X_0^{(n+270)\varepsilon} \end{cases}$$

e.g.



Γ: a strong ε chain contraction of C over $X - X_-$.

$$\Gamma_+ = \begin{cases} \Gamma & \text{over } X_+ - X_0^{(n+270)\varepsilon} \\ 0 & \text{over } X_+ \cap X_0^{(n+270)\varepsilon} \end{cases}$$



Γ_+ is a strong $\overset{90\varepsilon}{\cancel{\varepsilon}}$ chain contraction of C_+ over $X_+ - X_0^{(n+300)\varepsilon}$

- radius は ε. ✓
- $d_+\Gamma_+$, $\Gamma_+ d_+$ の radius は 91ε ✓

- $X_+ - X_0^{(n+300)\varepsilon}$ では.　$\Gamma_+ d_+ = \Gamma_+ \overset{\varepsilon}{\widehat{d}} = \Gamma d$.

$$d_+\Gamma_+ = d_+\underbrace{\Gamma}_{\varepsilon} = d\Gamma$$

$$\therefore d_+\Gamma_+ + \Gamma_+ d_+ = d\Gamma + \Gamma d \underset{2\varepsilon}{\sim} 1 .$$

- Γ_+^2 は？　over $X_+ - X_0^{(n+300)\varepsilon}$ で.

$$\Gamma_+^2 = \Gamma_+\underbrace{\Gamma}_{\varepsilon} = \Gamma\Gamma \underset{2\varepsilon}{\sim} 0 .$$ ✓

$$\therefore [C_+] \in Wh(X_+, X_+ \cap X_0^{(n+300)\varepsilon}, p_{X_+}, n, 90\varepsilon) .$$

C を fix したとき、C+ のとり方にはよらぬ. (by 2.5)

def　$\partial[C] = [C_+]$.

well-definedness

C, C'　strongly ε contractible over $X - X_-$

trivial　　　trivial

$$C \oplus D \oplus T \underset{40\varepsilon, \Sigma}{\cong} C' \oplus D' \oplus T'$$

n-dim, 40ε chain cx's on $\underline{(X_-)^{20\varepsilon}}$.

求める条件 1, 2 には無関係.

① $(C \oplus T)_+ = C_+ \oplus \underline{T(\cdots)}$
とできるから.　$\oplus T$ は無視してよい

② $(C \oplus D)_r(X_+ - X_0^{(n+90)\varepsilon}) = C_r(X_+ - X_0^{(n+90)\varepsilon})$ だから.
D もふくめてしまってよい.

∴ $C \underset{40\varepsilon, \Sigma}{\cong} C'$ と仮定し　1, 2 の条件をみたす C_+, C'_+ で
simple isomorphic なものをみつければよい.

$f : C \longrightarrow C'$ 40ε-simple iso.

$(C_+)_r = C_r(X_- - X_-^{(r+40)\varepsilon})$ とおく.

$g = f_{[X_- \cup X_0^{(n+160)\varepsilon}]}$ とおく。 これも 40ε-simple iso
(at each r)

$g = f$ over $X_- - X_0^{(n+200)\varepsilon}$

$g =$ gemetric over $X_- \cup X_0^{(n+80)\varepsilon}$.

$$\begin{tikzcd}
C_r \arrow[r, "g"] \arrow[d, "d"'] & C'_r = C''_r \arrow[d, dashed, "\overset{81\varepsilon}{gdg^{-1}}"] \\
C_{r-1} \arrow[r, "g"'] & C'_{r-1} = C''_{r-1}
\end{tikzcd}$$

$d'f \sim_{40\varepsilon} fd$.

$\underbrace{d'ff^{-1}}_{\sim_{81\varepsilon}\, d'} \sim_{80\varepsilon} fdf^{-1}$

C' と C'' は $\underline{X_+ - X_0^{(n+250)\varepsilon}}$ 上で.（d の ∼ をのぞいて）一致.

$gdg^{-1} = fdf^{-1} \sim_{81\varepsilon} d'$

$gdg^{-1}gdg^{-1}$

$\sim_{162\varepsilon} 0$.

C'' は 81ε chain map

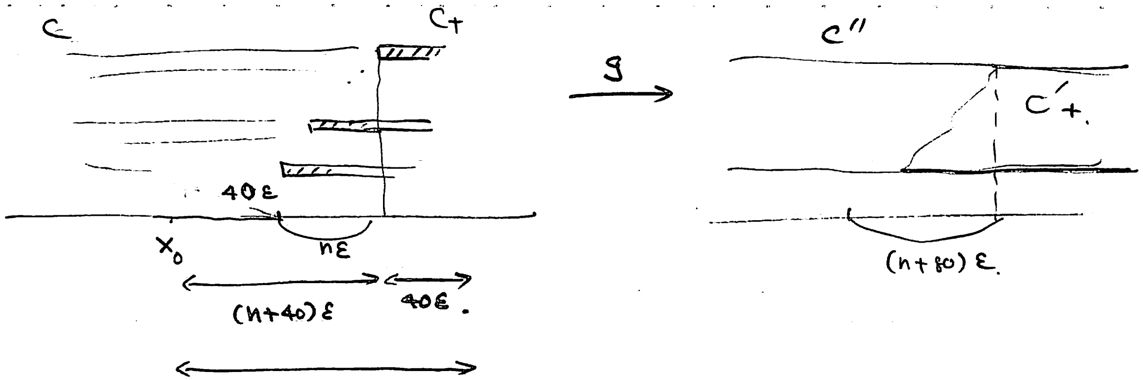
$g : C \xrightarrow{g} C''$, $\overset{g^{-1}}{\curvearrowleft}$

↴ $= \underbrace{gd}_{40\varepsilon}\underbrace{g^{-1}g}_{80\varepsilon} \sim_{120\varepsilon} gd =$ ↳.

↲ $= \underbrace{g^{-1}g}_{80\varepsilon}\underbrace{dg^{-1}}_{40\varepsilon} \sim_{120\varepsilon} dg^{-1} =$ ⬐

∴ g は 120ε chain map

∴ g は 120ε-simple isomorphism
(radius は 40ε).



$(n+80)\varepsilon$ で gは geometric

81ε chain map.

$$C_+ \xrightarrow{g} C'_+ \subset C''$$

120ε-simple iso.

$$(C'')_r \in X_+ - X_0^{(n+80)\ \varepsilon}) = (C'_+)_r\ (X_+ - X_0^{(n+80)\varepsilon})$$

$$d_{C''} = d_{C'_+} \quad \text{over} \quad X_+ - X_0^{(n+80)\varepsilon}$$

$$\therefore\ (C')_r\ (X_+ - X_0^{(n+80)\varepsilon}) = (C'_+)_r\ (X_+ - X_0^{(n+80)\varepsilon})$$

$$d_{C'} \underset{81\varepsilon}{\sim} d_{C'_+} \quad \text{over} \quad X_+ - X_0^{(n+250)\varepsilon}.$$

C'_+ の d を $\sim_{81\varepsilon}$ でかえればよい。

$$\begin{tikzcd} Wh(X, X_-; p_X, n, \varepsilon) \arrow[r, "exc"] \arrow[rr, bend left, "\partial_+"] & Wh(X_+, X_+ \cap X_0^{(n+300)\varepsilon}, p_{X_+}, n, 90\varepsilon) & {} \end{tikzcd}$$

$$\xrightarrow{\partial} \tilde{K}_0(X_+ \cap W, p_{X_+ \cap W}, n, \varepsilon') \longrightarrow K_0(W, p_W, n, \varepsilon')$$

$$W \supset \left(X_0^{(n+300)\varepsilon}\right)^{(4n+20)\cdot 90\varepsilon} \quad , \quad \boxed{\varepsilon' \geq (12n+70)\cdot 90\varepsilon}$$

1800
300

$(361n + 2100)\varepsilon$

$$\boxed{W \supset X_0^{400(n+10)\varepsilon}}$$

4.1

4.2 (Mayer-Vietoris)

(1).
$$\begin{tikzcd} Wh(X_-, p, n, \varepsilon) \oplus Wh(X_+, p, n, \varepsilon) \arrow[r, "(j_- \, j_+)"] & Wh(X, p_X, n, \varepsilon) \\ ([C_-], [C_+]) \arrow[r, mapsto] \arrow[u, phantom, "\in"] & 0 \arrow[u, phantom, "\in"] \end{tikzcd}$$

ふりわける

$$\exists f : C_- \oplus C_+ \oplus T' \longrightarrow T$$
$$\cong \quad \cancel{86}\,100\,\varepsilon, \Sigma$$

$$C_- \oplus C_+ \underset{\cong}{\xrightarrow{f}} T$$
$$\cancel{86}\,100\,\varepsilon, \Sigma$$

イキ
赤はとりやめ

$$f' = f_{[X_- \cup X_0^{\cancel{172}\,200\,\varepsilon}]}$$

- f' is ~~an 86~~ a 100 ε-simple iso.
- $f' = f$ over $X - (X_- \cup X_0^{\cancel{172}\,200\,\varepsilon})^{\cancel{86}\,100\,\varepsilon} \supset X - X_0^{\cancel{258}\,300\,\varepsilon}$
- f' = geometric over $(X_- \cup X_0^{\cancel{172\varepsilon}\,200})^{-\cancel{86}\,100\,\varepsilon} \supset X_- \cup X_0^{\cancel{86}\,100\,\varepsilon}$.

Tの boundary map を $f'(d_{C_-} \oplus d_{C_+})(f')^{-1}$ でかえる.

$\Rightarrow$ E ~~200ε~~ 200ε chain complex.

$$(\ \ )^2 = \underbrace{f'(d \oplus d)}_{200\varepsilon} \overbrace{f'^{-1} f'}^{200\varepsilon} \underbrace{(d \oplus d) f'^{-1}}_{200\varepsilon}$$

左ページへ ←

over $X - X_0^{300\varepsilon}$

$\cdot f^{-1} \cdot d_T$

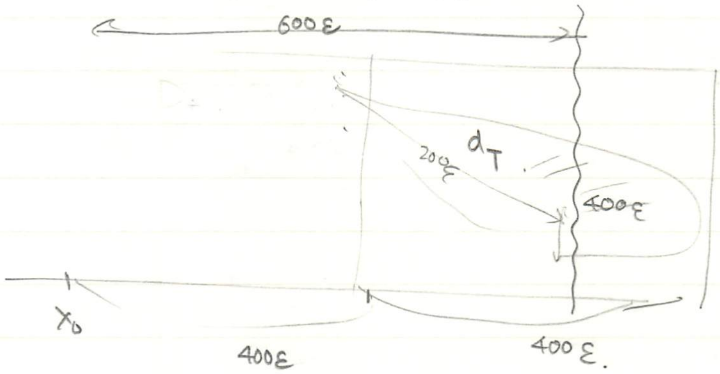
over $X_+ - X_0^{400\varepsilon}$

$$f'(d_{C_-}\oplus d_{C_+})(f')^{-1} = \overset{100\varepsilon\ \cancel{86\varepsilon}}{f'}\overset{86\varepsilon}{(d_{C_-}\oplus d_{C_+})f^{-1}}$$

$$\sim_{172\varepsilon} \underbrace{f' f^{-1}}_{86\varepsilon.}\ \underbrace{d_T}_{0.} \boxed{=} f f^{-1} d_T \sim_{2\cdot 86\varepsilon} d_T.$$

$\uparrow$ $4\cdot 86\varepsilon$

$X - X_0$



$$\begin{tikzcd} {} \arrow[r, "f'"] & {} \arrow[d, "f'(d_{C_-}\oplus d_{C_+})(f')^{-1}"] \\ & {} \end{tikzcd}$$

$$\rightharpoondown = \underset{86\varepsilon\cdot}{f'}\underset{\varepsilon.}{(d_{C_-}\oplus d_{C_+})}\underbrace{(f')^{-1}f'}_{2\cdot 86\varepsilon} \sim_{\cancel{300\varepsilon}\ 259} f'(d_{C_-}\oplus d_{C_+}) = \hookrightarrow$$ 258

$$\hookleftarrow = (f')^{-1} f' \cdot (d_{C_-}\oplus d_{C_+})(f')^{-1} \sim_{\cancel{300\varepsilon}\ 259} \downarrow$$

$(3\cdot 86+1)$

$\therefore f'$ は ~~300~~ ε- simple iso.

E は contractible.

$$\begin{tikzcd} (C_-\oplus C_+ \arrow[r, "f'", "r"'] & E_r \\ {} \arrow[u] \end{tikzcd}$$

3·8

$\exists \Gamma$: strong ε chain contraction

$$\Gamma^2 \sim_{2\varepsilon} 0 \qquad d\Gamma + \Gamma d \sim_{2\varepsilon} 1$$

$$\begin{array}{r} 260 \\ 86 \\ \hline 346 \end{array}$$

$$\underbrace{d(f'}_{\wr\ \cancel{259}\varepsilon}\overset{\varepsilon}{\Gamma}\underset{86}{(f')^{-1}}) + (f'\Gamma(f')^{-1})d \sim_{346\varepsilon} \overset{86\varepsilon}{f'}(d\Gamma + \Gamma d)\overset{86\varepsilon}{(f')^{-1}}$$

$$\sim_{(2\cdot 86+1)\varepsilon} f' f'^{-1} \sim_{2\cdot 86\varepsilon} 1$$

$$\overset{86\varepsilon}{\overbrace{f'}}\,\Gamma\,\overbrace{(f')^{-1}(f')}\,\Gamma\,\overset{86\varepsilon}{\overbrace{(f')^{-1}}}$$

$346\varepsilon = (3\cdot 86+1+86+1)$

$$\underset{(4\cdot 86+2)\varepsilon}{\sim} f'\,\Gamma^2\,(f')^{-1} \sim 0$$

$346 \div 2 = 173$

$\underline{\Gamma_E} = f'\Gamma(f')^{-1}$ は strong $\underline{\overset{173}{(2\cdot 86+1)}\varepsilon}$ chain contraction.

とおく

$E = D_- \oplus D_+ \oplus T''$

$d\Gamma_E + \Gamma_E d \underset{2\cdot 173\varepsilon}{\sim} 1$, $\Gamma_E^{\,2} \underset{2\cdot 173\varepsilon}{\sim} 0$.

C_-に対応する部分 $\nearrow D_-$

$D_+ \uparrow$ 1コ(つ)

$$\left.\begin{array}{ccc} D_+ & \underset{200\varepsilon}{\overset{1}{\xrightarrow[\simeq]{}}} & D_+ \\ \oplus & & \oplus \\ 0 & \underset{0}{\xrightarrow[\simeq]{}} & T'' \end{array}\right\} = E \underset{173\varepsilon}{\simeq} 0$$

$$\begin{array}{ccc} \oplus & & \oplus \\ 0 & \xrightarrow{\simeq\,(2\cdot 86+1)\varepsilon} & D_- \end{array}$$

C_-が ε contractible だから.

$D_+ \underset{373\varepsilon}{\simeq} 0$.

$373\times 3\,\varepsilon = 1119\,\varepsilon$

strongly 1200ε contractible

$$g : C_+ \underset{259\varepsilon,\,\Sigma}{\xrightarrow[\cong]{}} D_+ \oplus T''$$

$$\begin{array}{r} 259 \\ 86 \\ \hline 345 \end{array}$$

$$\underline{C_- \oplus D_+ \oplus T'' \xrightarrow{1_{C_-}\oplus g^{-1}} C_- \oplus C_+ \xrightarrow{\textcircled{f}} T}$$

$259\varepsilon, \Sigma$ $\quad$ $86\varepsilon, \Sigma$

$345\varepsilon, \Sigma$

on T'', $g^{-1} = (f')^{-1} = f^{-1}$.

∴ 上の合成は on T'で $\underset{172\varepsilon}{\sim} 1$

(2)

$$\begin{tikzcd}
Wh(X, p_X, n, \varepsilon) \arrow[r, "\partial_+"] & \tilde{K}_0(W, p_W, n, \varepsilon') \\
{[C]} \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] \arrow[d, red] & 0 \arrow[u, phantom, sloped, "\in"] \arrow[ddd, dash, bend left] \\
{[C] \in Wh(X, X_-, p_X, n, \varepsilon)} \arrow[d, red, "exc"] & \\
{[C_+] \in Wh(X_+, X_+ \cap X_0^{(n+300)\varepsilon}, p, n, 90\varepsilon)} \arrow[r, red, "\partial"] \arrow[d] & \tilde{K}_0(X_+ \cap W, p, n, \varepsilon') \arrow[d, red] \\
{[C_+] \in Wh(X_+ \cup W, X_0^{(n+300)\varepsilon}, p, n, 90\varepsilon)} \arrow[r, "\partial"] \arrow[r, bend right] \arrow[ur, phantom, "\circlearrowright"] & \tilde{K}_0(W, p_W, n, \varepsilon') \ni 0
\end{tikzcd}$$

C_+ : as before

(赤) が ∂_+

5.3 (2) により $\exists [\hat{C}_+] \in Wh(X_+ \cup W, p, n, \delta)$ s.t. $[\hat{C}_+] = [C_+]$ in $Wh(X_+ \cup W, W^{\delta} \cap (X_+ \cup W), p, n)$

$\delta = 6000(9n+34)\varepsilon'$.

同様に $\exists \hat{C}_-$ s.t. . . .

$[C] = [C_-] + [C_+]$ in $Wh(X, W^{\delta}, p, n, \delta)$

$\therefore [C] = [\hat{C}_-] + [\hat{C}_+]$ in $Wh(X, W^{\delta}, p, n, \delta)$.

5.3 (1)

$$\begin{tikzcd}
Wh(X, p, n, \delta) \arrow[r] & Wh(X, W^{\delta}, p, n, \delta) \\
{[C] - [\hat{C}_-] - [\hat{C}_+]} \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & 0 \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

$\therefore \exists [D] \in Wh(W^{2001\delta}, p, n, \hat{\varepsilon})$ s.t. $i_*[D] = [C] - [\hat{C}_-] - [\hat{C}_+]$ in $Wh(X, p, n, \hat{\varepsilon})$

$\hat{\varepsilon} = 9000(9n+34)\delta$.

$2001\delta = 2001 \cdot 6000(9n+34)\varepsilon' \leqq 13 \cdot 10^6 (9n+34)\varepsilon'$.

(3) $$\tilde{K}_0(W, p_W, n, \varepsilon') \xrightarrow{\begin{pmatrix} -i_- \\ i_+ \end{pmatrix}} \tilde{K}_0(X_- \cup W, p_{X_- \cup W}, n, \varepsilon') \oplus \tilde{K}_0(X_+ \cup W, \cdots)$$

$$\begin{tikzcd} \underset{[E, g]}{\Psi} \arrow[r, mapsto] & 0. \end{tikzcd}$$

- -

$f : C_- \oplus G \underset{8\gamma}{\xrightarrow{\simeq}} C_+ \oplus F$ on $p_{W'}$

$\mathcal{C}(f)$: strongly 72γ contractible. $\longrightarrow$ Γ とおく.

$[\widehat{\mathcal{C}(f)}] \in Wh(X, p_X, n, 72\gamma)$.

$\boxed{\mathcal{C}(f)}$

$$\begin{tikzcd} & (C_-)_n \oplus G_n \arrow[dl, "(-)^n f"'] \arrow[d] \\ (C_+)_n \oplus F_n \oplus \arrow[d] & (C_-)_{n-1} \oplus G'_{n-1} \arrow[dl, "\pm f"'] \arrow[d] \\ (C_+)_{n-1} \oplus F_{n-1} \oplus \arrow[d] & (C_-)_{n-2} \oplus G_{n-2} \arrow[dl, "\pm f"'] \arrow[d] \\ {} & {} \end{tikzcd}$$

$\widehat{\mathcal{C}(f)}$

radius 72γ.

$$\begin{tikzcd} (C_+)_n \oplus F_n \oplus \arrow[d] \arrow[drr, "\Gamma"] & (C_-)_{n-1} \oplus G_{n-1} \arrow[dl] \arrow[d] \arrow[dr, "\Gamma"] & \\ (C_+)_{n-1} \oplus F_{n-1} \oplus \arrow[d] & (C_-)_{n-2} \oplus G_{n-2} \oplus \arrow[dl] \arrow[d] & (C_-)_n \oplus G_n \\ \vdots & \vdots & \end{tikzcd}$$

$\therefore \widehat{\mathcal{C}(f)}_r (X_+ - W') = (C_+)_r (X_+ - W')$

$\hat{d}_r = d_{C_+}$ over $X_+ - (W')^{72\gamma}$.

$\delta = 72\gamma = 72 \cdot 180 (12n + 70) \varepsilon'$

$$\begin{tikzcd}
Wh(X, p_X, n, \overset{\substack{\delta \\ \|}}{\boxed{\cancel{72}45\gamma}}) \arrow[d] \arrow[dddd, bend right=60, "\partial_+"'] \\
Wh(X, X_-, p_X, n, \cancel{72}45\gamma) \arrow[d, "exc."'] \\
Wh(X_+, X_+ \cap X_0^{(n+300)\cancel{72}45\gamma}, p_{X_+}, n, 90 \cdot \cancel{72}45\gamma) \arrow[d, "\partial"] \\
\tilde{K}_0(X_+ \cap V, p, n, \delta') \arrow[d] \\
\tilde{K}_1(V, p_V, n, \delta')
\end{tikzcd}$$

$V \supset X_0^{\overset{400}{\cancel{300}}(n+10) \cdot \overset{\substack{\delta \\ \|}}{\boxed{72\gamma}}}$

$\delta' = (12n+70) \cdot 90 \cdot \underline{72\gamma}$

$= (12n+70) \cdot 90 \cdot 72 \cdot 180 \cdot$

$(12n+70) \varepsilon'$

§7

7.1

$p_X : M \longrightarrow X$ に対し $p'_X : M \times S^1 \xrightarrow{\text{proj.}} M \xrightarrow{p_X} X$ とおく.

tensor products.

$\mathbb{Z}[R]$ on M, $\mathbb{Z}[S]$ on N

$\mathbb{Z}[R] \otimes \mathbb{Z}[S]$: on $M \times N$ を次のように定める。

$$R : |R| \to M$$
$$S : |S| \to N$$

$$\begin{tikzcd} R \times S : |R| \times |S| \arrow[r] & M \times N \\ (|r|, |s|) \arrow[r, mapsto] \arrow[u, phantom, "\in" rotate=90] & ([r], [s]) \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

$$((|r|, |s|), ([r], [s])) \equiv r \otimes t$$

$$(r = (|r|, [r]), s = (|s|, [s]))$$

$$\mathbb{Z}[R] \otimes \mathbb{Z}[S] \overset{\text{def}}{=} \mathbb{Z}[R \times S]$$

(r, ρ, r') a path from $r \in R$ to $r' \in R'$

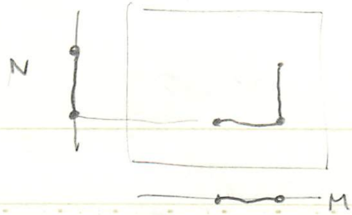
$[0, \tau] \to M$ both on M

(s, σ, s') a path from $s \in S$ to $s' \in S'$

$[0, \tau'] \to N$ both on N

$(r, \rho, r') \otimes (s, \sigma, s')$

$$\equiv (r \otimes s, \rho \otimes \sigma, r' \otimes s')$$



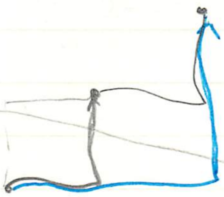
ここで $\rho\otimes\sigma : [0, \tau+\tau'] \longrightarrow M\times N$ は

$$\rho\otimes\sigma(x) = \begin{cases} (\rho(x), \sigma(0)) & \text{if } 0 \leq x \leq \tau \\ (\rho(\tau), \sigma(x-\tau)) & \text{if } \tau \leq x \leq \tau+\tau'. \end{cases}$$

と定める。

Tensor products of geometric morphisms are defined ---

$$(f'\otimes g')(f\otimes g) \sim f'f \otimes g'g$$



given $p_X : M \to X$

$p_Y : N \to Y$

use $p_X \times p_Y : M\times N \to X\times Y$

これによると・・

f, g : radius ε , f', g' : radius ε'.

$\Downarrow$ $\Downarrow$

$f\otimes g$: radius ε $f'\otimes g'$: radius ε'.

$$(f'\otimes g')(f\otimes g) \underset{\varepsilon+\varepsilon'}{\sim} f'f \otimes g'g .$$

$$\underbrace{[A,p]}_{\delta \text{ projective module}} \longmapsto f_p = (1-p)\otimes 1 + p\otimes z : A\otimes D \longrightarrow A\otimes D .$$

$\Rightarrow$ f_p is a δ isomorphism (via
$$\begin{tikzcd} M\times S^1 \arrow[r] \arrow[rr, bend left, "p'_X"] & X\times S^1 \arrow[r] & X \end{tikzcd}$$

$$((1-p)\otimes 1 + p\otimes z)((1-p)\otimes 1 + p\otimes z^{-1})$$

$$= ((1-p)\otimes 1)((1-p)\otimes 1) + ((1-p)\otimes 1)(p\otimes z^{-1})$$
$$+ (p\otimes z)((1-p)\otimes 1) + (p\otimes z)(p\otimes z^{-1})$$

$$\sim_\delta (1-p)^2\otimes 1 + (1-p)p\otimes z^{-1} + p(1-p)\otimes z + p^2\otimes zz^{-1} .$$

$$\sim_\delta (1-p)\otimes 1 + p\otimes zz^{-1}$$

$$\sim_\delta (1-p)\otimes 1 + p\otimes 1 = 1\otimes 1 .$$

$$f_{p\oplus 1_E} = f_p \oplus (1_E\otimes z)$$

$$\begin{tikzcd} (A\oplus E)\otimes D \arrow[r] \arrow[d, phantom, "\shortparallel"] & (A\oplus E)\otimes D \arrow[d, phantom, "\shortparallel"] \\ (A\otimes D)\oplus(E\otimes D) & (A\otimes D)\oplus(E\otimes D) \end{tikzcd}$$

$$\begin{tikzcd} g : (A, p) \arrow[r] & (A', p') \arrow[l, bend left, dashed, "g^{-1}"] \end{tikzcd}$$ δ isomorphism of δ proj. modules.

$$F : (A \otimes D) \oplus (A' \otimes D) \longrightarrow (A \otimes D) \oplus (A' \otimes D)$$

$$\begin{pmatrix} (1-p)\otimes 1 & g^{-1}\otimes 1 \\ g\otimes 1 & (1-p')\otimes 1 \end{pmatrix}$$ とおくと.

$$F^2 = \begin{pmatrix} (1-p)\otimes 1 & g^{-1}\otimes 1 \\ g\otimes 1 & (1-p')\otimes 1 \end{pmatrix}\begin{pmatrix} (1-p)\otimes 1 & g^{-1}\otimes 1 \\ g\otimes 1 & (1-p')\otimes 1 \end{pmatrix}$$

$$\sim_{2\delta} \begin{pmatrix} (1-p)^2\otimes 1 + g^{-1}g\otimes 1 & \overset{\sim 0}{\boxed{(1-p)g^{-1}\otimes 1}} + \overset{\sim 0}{\boxed{g^{-1}(1-p')\otimes 1}} \\ \underbrace{\boxed{g(1-p)\otimes 1}}_{\sim_\delta\, 0} + \underbrace{\boxed{(1-p')g\otimes 1}}_{\sim_\delta\, 0} & gg^{-1}\otimes 1 + (1-p')^2\otimes 1 \end{pmatrix}$$

$$\sim_{2\delta} \begin{pmatrix} (1-p)\otimes 1 + p\otimes 1 & 0 \\ 0 & p'\otimes 1 + (1-p')\otimes 1 \end{pmatrix} = 1$$

$$\therefore F^2 \sim_{2\delta} 1 .$$

$$(1\oplus f_{p'})F = \begin{pmatrix} 1 & 0 \\ 0 & (1-p')\otimes 1 + p'\otimes z \end{pmatrix}\begin{pmatrix} (1-p)\otimes 1 & g^{-1}\otimes 1 \\ g\otimes 1 & (1-p')\otimes 1 \end{pmatrix}$$

$$\sim_\delta \begin{pmatrix} (1-p)\otimes 1 & g^{-1}\otimes 1 \\ \cancel{(1-p')g\otimes 1} + p'g\otimes z & (1-p')^2\otimes 1 + p'(1-p')\otimes z \end{pmatrix}$$

$$\sim_\delta \begin{pmatrix} (1-p)\otimes 1 & g^{-1}\otimes 1 \\ g\otimes z & (1-p')\otimes 1 . \end{pmatrix}$$

$$F(f_p\oplus 1) = \begin{pmatrix} (1-p)\otimes 1 & g^{-1}\otimes 1 \\ g\otimes 1 & (1-p')\otimes 1 \end{pmatrix}\begin{pmatrix} (1-p)\otimes 1 + p\otimes z & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sim_\delta \begin{pmatrix} (1-p)^2\otimes 1 + \cancel{(1-p)p\otimes z} & g^{-1}\otimes 1 \\ \cancel{g(1-p)\otimes 1} + gp\otimes z & (1-p')\otimes 1 \end{pmatrix}$$

$\sim_\delta$.

7.1

$$\begin{tikzcd}
{[f_p]} & & {[A,p]} \arrow[ll, mapsto] \arrow[d, phantom, sloped, "\in"] & \text{としてよい.} \\
Wh(X, p'_x, n, 18\delta) \arrow[d] & & \widetilde{K}_0(X, p_x, n, \delta) \arrow[ll, "\overline{B}"'] \arrow[d] & \\
Wh(X, p'_x, n, \varepsilon) \arrow[rr, "B"'] \arrow[d, phantom, sloped, "\ni"] & & \widetilde{K}_0(X, p_x, n, K_n\varepsilon) \arrow[d, phantom, sloped, "\ni"] & \\
{[f_p]} \arrow[rr, mapsto, "?"'] & & {[A,p]} &
\end{tikzcd}$$

$[C] \in \operatorname{Ker} \overline{\partial}_+$

7.3.

$$\begin{tikzcd}
& \overset{[C]}{Wh(X,p_X,n,\varepsilon)} \arrow[r, "\partial_+"] \arrow[d, equal, "\frac{\varepsilon''}{18} \geqslant K_n \varepsilon ,"] \arrow[rr, bend left, "\overline{\partial}_+"] & \tilde{K}_0(W,p_W,n,\varepsilon''/18) \arrow[r, "\overline{B}"] \arrow[d, "W \supset X_0^{400(n+10)\varepsilon}"] & \overset{0}{Wh(W,p'_W,n,\varepsilon'')} \arrow[d, equal] \\
{} \arrow[r] & Wh(X,p_X,n,\varepsilon) \arrow[r, "\partial_+"] & \tilde{K}_0(W,p_W,n,K_n\varepsilon'') & Wh(W,p'_W,n,\varepsilon'') \arrow[l, "B"'] \\
& \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \\
& {[C]} \arrow[r, mapsto] & 0 &
\end{tikzcd}$$

∴ $[C]$ は 次のmapの像にはいる (after stabilization)

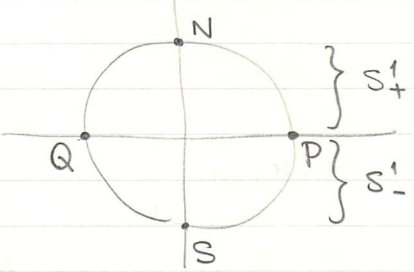
$$(j_- \; j_+) : Wh(X_- \cup \hat{W}, \cdots, \hat{\varepsilon}) \oplus Wh(X_+ \cup \hat{W}, \cdots, \hat{\varepsilon}) \longrightarrow Wh(X, p_X, n, \hat{\varepsilon})$$

$$\hat{W} = W^{13 \cdot 10^6 (9n+34) \cdot \overset{\varepsilon'}{\overset{\shortparallel}{\boxed{K_n \varepsilon''}}}}$$

$$\hat{\varepsilon} = 54 \cdot 10^6 (9n+34)^2 \cdot K_n \varepsilon''.$$

$\mathrm{Wh}(X\times S^1,\ p_X\times 1_{S^1},\ n,\ \varepsilon)$

S^1_+, S^1_- : 上(下)半分の円周.



$X\times(S^1;\ S^1_-,\ S^1_+)$ の Mayer-Vietoris 列をかいてみると‥
（但し 7.2 を用いて かんたんにしてある.）

$$\mathrm{Wh}(X, p_X, n, \varepsilon)\oplus \mathrm{Wh}(X, p_X, n, \varepsilon)\ \underset{\begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix}}{\xrightarrow{J}}\ \mathrm{Wh}(X, p_X, n, \varepsilon)\oplus \mathrm{Wh}(X, p_X, n, \varepsilon)$$

$$\begin{tikzcd}
\arrow[r, "(i_*\ i^+_*)"] & \mathrm{Wh}(X\times S^1, p_X\times 1, n, \varepsilon) \arrow[r, "\partial'_+"] \arrow[r, bend right, "B'"'] & \tilde{K}_0(X\times\{P\}, p_X, n, K_n\varepsilon) & \oplus & \tilde{K}_0(X\times\{Q\}, p_X, n, K_n\varepsilon)
\end{tikzcd}$$

$$\underset{\begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix}}{\xrightarrow{J'}}\ \tilde{K}_0(\underset{\substack{\shortparallel\\ X\times\{S\}}}{X}, p_X, n, K_n\varepsilon)\oplus \tilde{K}_0(\underset{\substack{\shortparallel\\ X\times\{N\}}}{X}, p_X, n, K_n, \varepsilon)$$

◎ $\underset{\substack{\shortparallel\\ \delta}}{\boxed{54\cdot 10^6\cdot(9n+34)^2 K_n\varepsilon}} \ll 1$ とする。
δ とおく。

$$\begin{tikzcd}
B' : \mathrm{Wh}(X\times S^1, p_X\times 1, n, \varepsilon) \arrow[r] & \tilde{K}_0(\overset{\substack{X\times\{P\}\\ \shortparallel}}{X}, p_X, n, K_n\varepsilon) \\
\mathrm{Wh}(X, p_X, n, \varepsilon) \arrow[u, "i^+_*"] \arrow[ur, "0"']
\end{tikzcd}$$
とおく。

injective である。

$(p_M)_* : \mathrm{Wh}(X\times S^1, p_X\times 1, n, \varepsilon) \to \mathrm{Wh}(X, p_X, n, \varepsilon)$.

$p_M : M\times S^1 \underset{\text{proj.}}{\longrightarrow} M$.

$(p_M)_*(i^+)_* = 1$ だから

p.45. (1).

$$\begin{tikzcd}
Wh(X\times\{S\}) \oplus Wh(X\times\{N\}) \arrow[d, bend right, "(i^-)_*"'] \arrow[d] \arrow[d, bend left, "(i^+)_*"] & \\
Wh(X\times S^1) \arrow[u, bend right=60, "(p_M)_*"'] \arrow[d, "\partial'_+"] \arrow[dr, "B'"] & \\
\tilde{K}_0(X\times\{Q\}) \oplus \tilde{K}_0(X\times\{P\}) \arrow[r] \arrow[d, "\begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix}"] & \tilde{K}_0(X\times\{P\}) \\
\tilde{K}_0(X\times\{S\}) \oplus \tilde{K}_0(X\times\{N\}) &
\end{tikzcd}$$

$$\begin{tikzcd}
& x \arrow[d, mapsto] \arrow[dr] & \\
\text{実は}\,0 & (*,0) \arrow[l, Rightarrow] \arrow[d, mapsto, "J'"] & 0 \\
& ? &
\end{tikzcd}$$

$$? = \begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}*\\ 0\end{pmatrix} = \begin{pmatrix}-*\\ *\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \quad \therefore * = 0$$

∴ x は上からくる!!
<u>stable に</u>

$$\begin{tikzcd}
(\alpha,\beta) \in Wh(X)\oplus Wh(X). \arrow[d, mapsto] \\
x.
\end{tikzcd}$$

ならば $(0, \alpha+\beta) \longmapsto x$

$\alpha+\beta \longmapsto x$ としてよい. □

* ここで、$\delta = 54\cdot 10^6 (9n+34)^2 K_n \varepsilon$ が必要.

(2). stableに B'は onto.

$$\begin{tikzcd}
Wh(X \times S^1, \varepsilon) \arrow[r, "B'"] \arrow[d] & \tilde{K}_0(X, k_n \varepsilon)_{\circ x} \arrow[d] \\
Wh(X \times S^1, \delta) \arrow[r, "B'"'] & \tilde{K}_0(X, k_n \delta)
\end{tikzcd}$$

$$J' \begin{pmatrix} \overset{\text{proj}}{x} \\ -x \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ -x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{tikzcd}
\bullet \arrow[d, dashed] & \\
(-x, x) \arrow[r, mapsto] \arrow[d, "J'"] & x \\
0 &
\end{tikzcd}$$

ここでは $\underline{\delta = 72 \cdot 180 (12n + 70) k_n \varepsilon}$. が必要.

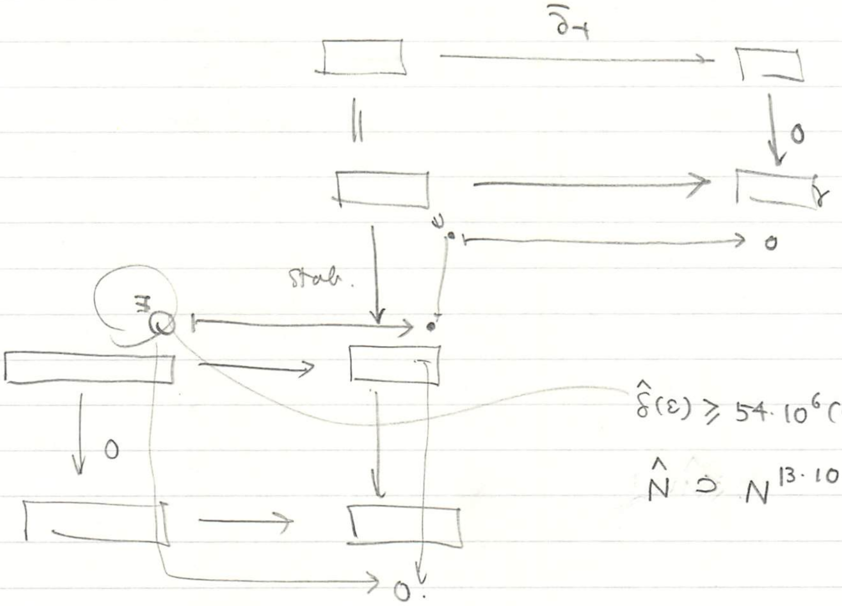
これと. $\Uparrow$

$\boxed{54 \cdot 10^6 (9n + 34)^2 k_n \varepsilon.}$ これを用いる.

§8

$\bar{\partial}_+$ が def できるために‥

- $\delta^0(\gamma) \geqslant 18K_n \delta^L(\varepsilon)$
- $N \supset L_0^{400(n+10)\delta^L(\varepsilon)}$



$\hat{\delta}(\varepsilon) \geqslant 54 \cdot 10^6 (9n+34)^2 K_n \gamma$

$\hat{N} \supset N^{13 \cdot 10^6 (9n+34) K_n \gamma}$

Subdivision（cell の X における像の直径 < ε.）

$f : K' \longrightarrow L'$

$\Uparrow$

$f : K \longrightarrow L$　　$p_X^{-1}(\varepsilon)$- equivalence

"diameter" ではないか.

＊ K', L' は やはり transverse CW 複体.

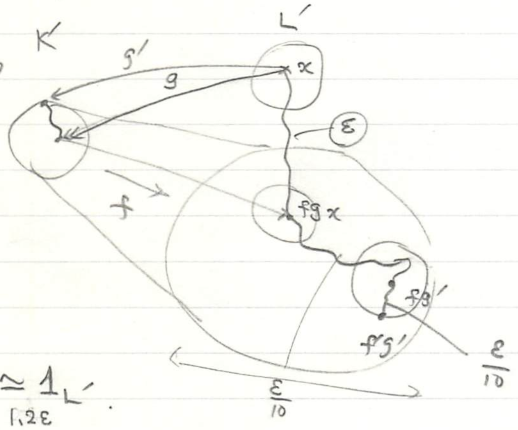
f を t-cellular に近似しよう。

L' は "saturated" in the sense of Quinn [16] ; therefore,

(2-1) Subdivide K and L if necessary, and assume that the diameter of the image in X of each cell of K and L via $p_X f$ and p_X is smaller than ε/10. L is "saturated" in the sense of Quinn [16]. Therefore f is $p_X^{-1}(\varepsilon)$-homotopic to a t-cellular $p_X^{-1}(2\varepsilon)$-equivalence $f' : K' \to L'$. Let $g' : L' \to K'$ be a t-cellular $p_X^{-1}(2\varepsilon)$-homotopy inverse.

$$F = f'_{\%} : \underbrace{f'_{\#} C(K)}_{\uparrow\ \varepsilon \text{ chain complex}} \xrightarrow{2\varepsilon \text{ chain map}} \underset{\uparrow\ \frac{\varepsilon}{10} \text{ chain cx}}{C(L)}$$

$$G : \underset{\text{geom. } 2\varepsilon \text{ iso}}{C(L)} \xrightarrow{\cong} f'_{\#}\, g'_{\#}\, C(L) \xrightarrow[\uparrow\ 2\varepsilon \text{ chain map}]{f'_{\#}(g'_{\%})} f'_{\#} C(K)$$



$$\begin{array}{c} fg \simeq_{\varepsilon} 1_{L'} \\ \wr|_{\frac{\varepsilon}{10}} \\ fg' \\ \wr|_{\frac{\varepsilon}{10}} \\ f'g' \end{array}$$

$\therefore\ f'g' \underset{1.2\varepsilon}{\simeq} 1_{L'}$.

このホモトピーは t-cellular ではない!!

$\Downarrow$

$f'g' \underset{1.3\varepsilon}{\simeq} 1_{L'}$

こんどは t-cellular.

$\therefore\ f'g' \overset{\text{t-cell}}{\underset{2\varepsilon}{\simeq}} 1$.

同様に.

$gf : K \longrightarrow K \quad \simeq 1_{K'}$

$\Downarrow$

$g'f' \underset{1.3\varepsilon}{\simeq} 1_K$ (t-cellular)

via $p_X f$

これを $p_X f'$ ではかると.

$g'f' \underset{1.4\varepsilon}{\simeq} 1_{K'}$

$\therefore$ $p_X^{-1}(2\varepsilon)$-equivalence !!

合成 FG について・・・

$$\begin{tikzcd}
f'_\# g'_\# C(L) \arrow[r, "f'_\#(g'_\%)"] \arrow[d, equal] & f'_\# C(K) \arrow[r, "\overset{F}{\overset{\Downarrow}{f'_\%}}"] \arrow[d, phantom, "\sim"] & C(L) \arrow[d, equal] \\
(f'g')_\# C(L) \arrow[rr, "(f'g')_\%"] \arrow[d, "\cong"] & {} \arrow[d, phantom, "\simeq"] & C(L) \arrow[d, equal] \\
C(L) \arrow[rr, "id"'] \arrow[uur, bend left=60, "G"] & & C(L)
\end{tikzcd}$$

"homotopy" of germ. morphisms.

<u>2ε chain htpy</u>

同様に

$$\begin{tikzcd}
g'_{\#} f'_{\#} C(K) \arrow[r, "g'_{\#}(f'_{\%})"] \arrow[d, equal] & g'_{\#} C(L) \arrow[r, "g'_{\%}"] \arrow[d, phantom, "\sim"] & C(K) \arrow[d, equal] \\
(g'f')_{\#} C(K) \arrow[rr, "(g'f')_{\%}"] & {} \arrow[d, phantom, "\simeq"] & C(K) \arrow[d, equal] \\
C(K) \arrow[u, "\simeq"] \arrow[rr, "1"] & & C(K)
\end{tikzcd}$$

これに $f'_{\#}$ をつける.

$$\begin{tikzcd}
f'_{\#} C(K) \arrow[r, "f'_{\%} = F"] \arrow[d, "f'g' \simeq 1 \rightarrow \simeq"'] & C(L) \arrow[d, "\simeq \leftarrow f'g' \simeq 1"] \arrow[ddr, bend left, "G"] & \\
f'_{\#} g'_{\#} f'_{\#} C(K) \arrow[r, "f'_{\#} g'_{\#}(f'_{\%})"] \arrow[d, equal] & f'_{\#} g'_{\#} C(L) \arrow[r, "f'_{\#}(g'_{\%})"] \arrow[d, phantom, "\sim"] & f'_{\#} C(K) \arrow[d, equal] \\
f'_{\#}(g'f')_{\#} C(K) \arrow[rr] \arrow[d, "g'f' \simeq 1 \rightarrow \simeq"'] & {} \arrow[d, phantom, "\simeq"] & f'_{\#} C(K) \arrow[d, equal] \\
f'_{\#} C(K) \arrow[rr] \arrow[uuu, bend left, "\lambda"] & {} & f'_{\#} C(K)
\end{tikzcd}$$

$$GF\lambda \underset{10\varepsilon}{\simeq} 1$$

$\underbrace{f'_{\%}}$

$$F\lambda = f'_{\%}\lambda \simeq f'_{\%}$$