ON FIXED POINT DATA OF SMOOTH ACTIONS ON SPHERES

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We report results obtained jointly with K. Pawałowski.

There are two fundamental questions about **smooth actions** on manifolds. Let G be a finite group and M a manifold.

Question 1. Which manifolds F can be the G-fixed point sets of G-actions on M, i.e. $M^G = F$?

Question 2. Which G-vector bundles ν over F can be the G-tubular neighborhoods (i.e. G-normal bundles) of $F = M^G$ in M?

If ν can be realized as a subset of M in the way above, we say that (F, ν) occurs as the **G-fixed point data** in M. If for a real G-module W with $W^G = 0$, $(F, \nu \oplus \varepsilon_M^W)$ occurs as the G-fixed point data in M then we say that (F, ν) **stably** occurs as the G-fixed point data in M. These questions were studied by B. Oliver [O2] in the case where G is not of prime power order and M is a disk or a Euclidean space. The topic of the current talk is the case where G is an Oliver group and M is a sphere.

Let G be a finite group not of prime power order. A G-action on M is called **$\mathcal{P}$-proper** if $M^P \supsetneq M^G$ for any Sylow subgroup P of G. There are necessary conditions for (F, ν) to stably occur as the G-fixed point data of a $\mathcal{P}$-proper G-action on a sphere.

(F1) (Oliver Condition) $\chi(F) \equiv \chi(M) \mod n_G$ (where n_G is the integer called Oliver's number [O1]).

(B1) (Product Bundle Condition) $\tau_F \oplus \nu = 0$ in $\widetilde{KO}(F)$.

(B2) (Smith Condition) For each prime p and any Sylow p-subgroup P of G, $\tau_F \oplus \nu = 0$ in $\widetilde{KO}_P(F)_{(p)}$.

By Oliver [O2], Conditions (F1), (B1) and (B2) are also necessary–sufficient conditions for (F, ν) to stably occur as the G-fixed point data in a disk. By [O1], n_G is equal to 1 if and only if there are no normal series $P \triangleleft H \triangleleft G$ such that $|P| = p^s$, H/P is cyclic, and $|G/H| = q^t$ ($s,\ t \geq 0$). A group G with $n_G = 1$ is called an **Oliver group**. Clearly any nonsolvable group is an Oliver group. A nilpotent group is an Oliver group if and only if it has at least three noncyclic Sylow subgroups. In the case where G is an Oliver group, Condition (F1) provides no restriction.

We begin the preparation for our sufficient conditions. For a finite group G and a prime p, let G^p denote the minimal normal subgroup of G such that G/G^p is of

p-power order (possibly $G^p = G$). Let $\mathcal{L}(G)$ denote the set of all subgroups H of G such that $H \supseteq G^p$ for some prime p. Let $\mathcal{P}(G)$ denote the set of all subgroups P of G such that $|P|$ is a prime power (possibly $|P| = 1$). A G-action on M is said to be $(\mathcal{P}, \mathcal{L})$-**proper** if the action is $\mathcal{P}$-proper and if any connected component of X^H $(H \in \mathcal{L}(G))$ does not contain a connected component of M^G as a proper subset. If G is an Oliver group then the G-action on

$$V(G) = (\mathbb{R}[G] - \mathbb{R}) - \bigoplus_{p \backslash |G|} (\mathbb{R}[G/G^p] - \mathbb{R})$$

is $(\mathcal{P}, \mathcal{L})$-proper ([LM]). A finite group G is said to be **admissible** if there is a real G-module V such that $\dim V^P > 2 \dim V^H$ for any $P \in \mathcal{P}(G)$ and any $H \leq G$ with $H \supsetneq P$, and $\dim V^H = 0$ for any $H \in \mathcal{L}(G)$.

Theorem (M.M.–M. Yanagihara [MY1–2])**.** *Let G be an Oliver group. If $G^2 = G$ or $G^p \neq G$ for at least 2 distinct odd primes then G is admissible. In particular, an Oliver group G is admissible in each case: G is nilpotent; G is perfect.*

The symmetric group of degree 5 is not admissible.

K. H. Dovermann–M. Herzog recently proved that S_n $(n \geqq 6)$ are admissible.

Our main result is:

Theorem A. *Let G be an admissible Oliver group* (*resp. an Oliver group*) . *Let F be a closed manifold* (*resp. a finite discrete space*) *and let ν be a real G-vector bundle over F such that $\dim \nu^H = 0$ whenever $H \in \mathcal{L}(G)$. Then the following* (1)–(3) *are equivalent:*

(1) *(F, ν) stably occurs as the G-fixed point data of a $(\mathcal{P}, \mathcal{L})$-proper G-action on a sphere.*

(3) *(F, ν) stably occurs as the G-fixed point data in a disk.*

(3) *$\tau_M \oplus \nu$ satisfies* (B1)–(B2).

A finite group not of prime power order belongs to exactly one of the following six classes ([O2]):

$\mathcal{A}$: G has a dihedral subquotient of order $2n$ for a composite integer n.

$\mathcal{B}$: $G \notin \mathcal{A}$ and G has a composite order element conjugate to its inverse.

$\mathcal{C}$: $G \notin \mathcal{A} \cup \mathcal{B}$, G has a composite order element and the Sylow 2-subgroups are not normal in G.

$\mathcal{C}_2$: G has a composite order element and the Sylow 2-subgroup is normal in G.

$\mathcal{D}$: G has no elements of composite order and the Sylow 2-subgroups are not normal in G.

$\mathcal{D}_2$: G has no elements of composite order and the Sylow 2-subgroup is normal in G.

Corollary B. *Let G be a nontrivial perfect group and F a closed manifold. Then F occurs as the G-fixed point set of a $\mathcal{P}$-proper G-action on a sphere if and only if F occurs as the G-fixed point set in a disk* (*in other words,*

$G \in \mathcal{A}$: *there is no restriction.*

$G \in \mathcal{B}$: $c_{\mathbb{R}}([\tau_F]) \in c_{\mathbb{H}}(\widetilde{KSp}(F)) + \mathrm{Tor}(\widetilde{K}(F))$

$G \in \mathcal{C}$: $[\tau_F] \in r_{\mathbb{C}}(\widetilde{K}(F)) + \mathrm{Tor}(\widetilde{KO}(F))$

$G \in \mathcal{D}$: $[\tau_F] \in \mathrm{Tor}(\widetilde{KO}(F))$.)

Theorem C. *Let G be a nilpotent Oliver group and F a closed manifold. Then the following* (1)–(3) *are equivalent.*

(1) *F occurs as the G-fixed point set of a $\mathcal{P}$-proper G-action on a sphere.*

(2) *τ_F is stably complex.*

(3) *F occurs as the G-fixed point set of a G-action on a disk.*

Our basic methods are:

(1) An extension of the method of equivariant bundles in [O2] (with modifications).

(2) The equivariant thickening of [P].

(3) The equivariant surgery results of [M1–2].

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