

THE MATSUMOTO TRIPLING FOR COMPACT SIMPLY
CONNECTED 4-MANIFOLDS

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1. **Introduction.** Let W be an oriented simply connected 4-manifold and let x_1, x_2, x_3 be three elements of $H_2(W; \mathbf{Z})$ with mutual intersection numbers $x_i \cdot x_j = 0$ ($i \neq j$). In [3], by analysing Whitney's tricks for intersections of immersed 2-spheres representing x_1, x_2, x_3 in W , Y. Matsumoto introduced a number $\langle x_1, x_2, x_3 \rangle$ as an element of \mathbf{Z} modulo an ideal $I = I\{x_1, x_2, x_3\} = \{x_1 \cdot u_1 + x_2 \cdot u_2 + x_3 \cdot u_3 \mid u_1, u_2, u_3 \in H_2(W; \mathbf{Z})\}$. The tripling \langle, \rangle will be referred to as the Matsumoto tripling and the ideal I will be called the intersection ideal of $\{x_1, x_2, x_3\}$.

It has been shown that x_1, x_2, x_3 can be realized by mutually disjoint immersed 2-spheres if and only if $\langle x_1, x_2, x_3 \rangle = 0$, (for the "only if" part see [3] and for the "if" part see [7]). If W is closed, $\langle x_1, x_2, x_3 \rangle$ always vanishes because of the Poincaré duality.

Suppose that the boundary $M = \partial W$ of W is non-empty. For an integer d , a homology class $x \in H_2(W; \mathbf{Z})$ has a mod d boundary reduction $y \in H_2(M; \mathbf{Z}_d)$, if $i_* y = x \bmod d$ for the inclusion map $i: M \rightarrow W$.

Our aim in this paper is to prove the following;

THEOREM. *Let (W, M) be a compact oriented simply connected 4-manifold with non-empty boundary $\partial W = M$. Suppose that we are given three elements $x_1, x_2, x_3 \in H_2(W; \mathbf{Z})$ with mutual intersection numbers zero and with the intersection ideal $I = (d)$, $d \in \mathbf{Z}$. Then each element x_i has a unique mod d boundary reduction y_i , $i = 1, 2, 3$ and the following equality holds;*

$$\langle x_1, x_2, x_3 \rangle = -(y_1^* \cup y_2^* \cup y_3^*) \cap [M] \quad \text{in } \mathbf{Z}_d,$$

where $y_i^* \in H^1(M; \mathbf{Z}_d)$ is the Poincaré dual of y_i in M .

Thus the Matsumoto tripling \langle, \rangle is completely determined by the multiple cup product of the mod d boundary reductions in the boundary.

An implication of Theorem is

COROLLARY 1 (Invariance of Matsumoto tripling). *Let (W, M) (W', M') and $x_1, x_2, x_3 \in H_2(W; \mathbf{Z})$ be 4-manifolds with boundary and*

homology classes as in Theorem. If $f: (W, M) \rightarrow (W', M')$ is a map such that the restriction $g = f|_M: M \rightarrow M'$ is of degree one, i.e., $g_*[M] = [M']$, then we have $I\{x_1, x_2, x_3\} \supset I\{f_*x_1, f_*x_2, f_*x_3\}$ and

$$\langle f_*x_1, f_*x_2, f_*x_3 \rangle' \equiv \langle x_1, x_2, x_3 \rangle \quad \text{in } \mathbf{Z}/I\{x_1, x_2, x_3\},$$

where $\langle f_*x_1, f_*x_2, f_*x_3 \rangle'$ is the reduction of $\langle f_*x_1, f_*x_2, f_*x_3 \rangle$ in $\mathbf{Z}/I\{x_1, x_2, x_3\}$.

The proof of Theorem will be divided into two cases; $I = (0)$ (§ 2) and $I \neq (0)$ (§ 3). In § 4, we shall give some applications of Theorem as well as the proof of Corollary 1.

2. The proof of Theorem; part 1. Since the homomorphism $j_*: H_2(W; \mathbf{Z}) \rightarrow H_2(W, M; \mathbf{Z})$ induced by the inclusion map $j: W \rightarrow (W, M)$ is represented by the intersection matrix for $H_2(W)$, it follows that a homology class $x \in H_2(W; \mathbf{Z})$ has a mod d boundary reduction if and only if the ideal (d) contains the intersection ideal $I\{x\} = \{x \cdot u \mid u \in H_2(W; \mathbf{Z})\}$. Hence for an integer d , each $x_i \in H_2(W; \mathbf{Z})$, $i = 1, 2, 3$, has a mod d boundary reduction y_i if and only if (d) contains the intersection ideal $I = I\{x_1, x_2, x_3\}$. Since W is simply connected, we have a short exact sequence:

$$H_3(W, M; \mathbf{Z}_d) = 0 \longrightarrow H_2(M; \mathbf{Z}_d) \xrightarrow{i_*} H_2(W; \mathbf{Z}_d) \xrightarrow{j_*} H_2(W, M; \mathbf{Z}_d).$$

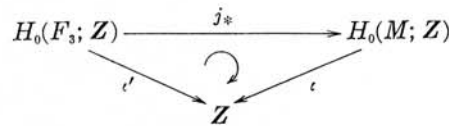
Thus the reduction y_i is unique. In particular, if $I = (0)$, i.e., $j_*x = 0$, then each x_i has a unique integral reduction y_i .

In this section, we shall prove Theorem in this special case $I = (0)$.

Represent y_1, y_2, y_3 by smoothly embedded oriented surfaces $\tilde{F}_1, \tilde{F}_2, \tilde{F}_3$. We may assume that those surfaces are in general position. Let $j: \tilde{F}_3 \rightarrow M$ be the inclusion map. Then

$$\begin{aligned} (y_1^* \cup y_2^* \cup y_3^*)[M] &= \iota((y_1^* \cup y_2^* \cup y_3^*) \cap [M]) = \iota'((j^*y_1^* \cup j^*y_2^*) \cap [\tilde{F}_3]) \\ &= [\gamma_1] \cdot [\gamma_2] \end{aligned}$$

where ι, ι' are augmentations, $\gamma_i = \tilde{F}_i \cap \tilde{F}_3$, $i = 1, 2$, $[\gamma_i] \in H_1(\tilde{F}_3; \mathbf{Z})$ is the homology class represented by γ_i (see the diagram below).



Let

$$\begin{aligned} \tilde{C}_1 &= \{\tilde{\alpha}'_1, \tilde{\beta}'_1; \dots; \tilde{\alpha}'_r, \tilde{\beta}'_r\} \\ \tilde{C}_2 &= \{\tilde{\alpha}''_1, \tilde{\beta}''_1; \dots; \tilde{\alpha}''_s, \tilde{\beta}''_s\} \end{aligned}$$

$$\tilde{C}_3 = \{\tilde{\alpha}_1, \tilde{\beta}_1; \dots; \tilde{\alpha}_t, \tilde{\beta}_t\}$$

be the canonical systems of simple closed curves of $\tilde{F}_1, \tilde{F}_2, \tilde{F}_3$, respectively, where r (resp. s, t) is the genus of \tilde{F}_1 (resp. \tilde{F}_2, \tilde{F}_3). We can choose systems so that two curves belonging to distinct systems \tilde{C}_i, \tilde{C}_j have no point in common. Now,

$$[\gamma_1] \cdot [\gamma_2] = \sum_{k=1}^t (a_k^1 \cdot b_k^2 - a_k^2 \cdot b_k^1)$$

where

$$[\gamma_i] = \sum_{k=1}^t (a_k^i [\tilde{\alpha}_k] + b_k^i [\tilde{\beta}_k]), \quad a_k^i, b_k^i \in \mathbb{Z}.$$

Let $M \times [0, 1] \subset W$ be a collar neighborhood of $\partial W = M = M \times \{0\}$ in W . The classes $i_* y_i$ can be represented by the surface $F_i = \tilde{F}_i \times (4 - i)/4 \subset \text{int } W$. Translating the systems $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3$ along the collar, we obtain the canonical systems of simple closed curves of F_1, F_2, F_3 :

$$\begin{aligned} C_1 &= \{\alpha'_1, \beta'_1; \dots; \alpha'_r, \beta'_r\} \\ C_2 &= \{\alpha''_1, \beta''_1; \dots; \alpha''_s, \beta''_s\} \\ C_3 &= \{\alpha_1, \beta_1; \dots; \alpha_t, \beta_t\}. \end{aligned}$$

As $W - M \times [0, 1)$ is simply connected, the curves $\tilde{\alpha}'_i \times 1, \tilde{\alpha}''_j \times 1, \tilde{\alpha}_k \times 1$ in $M \times \{1\}$ bound immersed disks d'_i, d''_j, d_k in $W - M \times [0, 1)$. We set

$$\begin{aligned} D'_i &= \tilde{\alpha}'_i \times [3/4, 1] \cup d'_i \\ D''_j &= \tilde{\alpha}''_j \times [2/4, 1] \cup d''_j \\ D_k &= \tilde{\alpha}_k \times [1/4, 1] \cup d_k. \end{aligned}$$

By spinning $D'_i (D''_j, D_k)$ around $\alpha'_i (\alpha''_j, \alpha_k)$, if necessary, we may assume that the normal bundle $\nu(\alpha'_i \hookrightarrow F_1)$ extends to a sub-bundle of $\nu(D'_i \rightarrow W)$, etc. Using these disks D'_i, D''_j, D_k , perform surgery on F_1, F_2, F_3 in $\text{int } W$, and we obtain immersed 2-spheres S_1, S_2, S_3 representing x_1, x_2, x_3 . The construction is as follows: using the sub-bundle of $\nu(D'_i \rightarrow W)$ (resp. $\nu(D''_j \rightarrow W), \nu(D_k \rightarrow W)$) mentioned above, we obtain an immersion f'_i (resp. f''_j, f_k): $D^2 \times [-1, 1] \rightarrow W$, such that $f'_i(D^2 \times \{0\}) = D'_i$ (resp. $f''_j(D^2 \times \{0\}) = D''_j, f_k(D^2 \times \{0\}) = D_k$) and $N'_i = f'_i(\partial D^2 \times [-1, 1])$ (resp. $N''_j = f''_j(\partial D^2 \times [-1, 1]), N_k = f_k(\partial D^2 \times [-1, 1])$) is a small tubular neighborhood of α'_i (resp. α''_j, α_k) in F_1 (resp. F_2, F_3). Then

$$\begin{aligned} S_1 &= \left(F_1 - \bigcup_{i=1}^r N'_i \right) \cup \bigcup_{i=1}^r f'_i(D^2 \times \{\pm 1\}) \\ S_2 &= \left(F_2 - \bigcup_{j=1}^s N''_j \right) \cup \bigcup_{j=1}^s f''_j(D^2 \times \{\pm 1\}) \\ S_3 &= \left(F_3 - \bigcup_{k=1}^t N_k \right) \cup \bigcup_{k=1}^t f_k(D^2 \times \{\pm 1\}). \end{aligned}$$

Now we shall construct Whitney disks $\Delta^{(i,j)}$'s and compute $\langle x_1, x_2, x_3 \rangle$.

(1) Whitney disks of type 1. Corresponding to an intersection point of α_k and the double curve γ_1 (resp. γ_2), there occur two intersection points, p and q , of S_3 and S_1 (resp. S_2) with opposite signs. We draw two arcs $\gamma_1^{(3,1)}, \gamma_3^{(3,1)}$ (resp. $\gamma_2^{(2,3)}, \gamma_3^{(2,3)}$) connecting p and q on S_1 (resp. S_2, S_3). Let the arc $\gamma_1^{(3,1)}$ (resp. $\gamma_2^{(2,3)}$) lie in $f_k(D^2 \times [-1, 1])$, and if D_k has not been spun around α_k , let the arc $\gamma_3^{(3,1)}$ (resp. $\gamma_3^{(2,3)}$) go straight down to reach the height of $1/4$ and run parallel with β_k on $F_3 - \text{int } N_k$ to the other component of ∂N_k and go straight up to the end point.

Now, if D_k has not been spun, the Whitney disk is;

$$\Delta^{(3,1)} = (\gamma_1^{(3,1)} \times [3/4, 1]) \cup ((\gamma_3^{(3,1)} \cap F_3) \times [1/4, 1]) \cup (\text{an immersed 2-disk in } W - M \times [0, 1])$$

$$\Delta^{(2,3)} = (\gamma_2^{(2,3)} \times [2/4, 1]) \cup ((\gamma_3^{(2,3)} \cap F_3) \times [1/4, 1]) \cup (\text{an immersed 2-disk in } W - M \times [0, 1]).$$

If D_k has been spun around α_k , change this disk by homotopy to obtain the desired Whitney disk, keeping the part of level higher than $3/8$ unchanged. Similarly there exists a Whitney disk $\Delta^{(1,2)}$ corresponding to an intersection point of $\tilde{\alpha}_j'$ and $\tilde{\gamma}$, where $\tilde{\gamma} = \tilde{F}_1 \cap \tilde{F}_2$ is the double curve on F_2 . We shall call these disks Whitney disks of type 1.

We orient W as follows:

$$[W] = [M] \times [n],$$

where n is the outward normal vector and $[]$ is the orientation. Let the sign of the intersection point of α_k and γ_1 be $\varepsilon (= \pm 1)$, i.e., $[\alpha_k] \times [\gamma_1] = \varepsilon [F_3]$. This is equivalent to saying that $[\gamma_1] = \varepsilon [\beta_k]$ near the intersection point. The curves γ_1, γ_2 are oriented as follows (see Figure 1):

$$[S_1] = [\gamma_1] \times [v]$$

$$[S_2] = [\gamma_2] \times [v],$$

where v is a normal vector field on F_3 in M such that $[F_3] \times [v] = [M]$. Let p be the intersection point of S_1 and S_3 on the side where $[\beta_k]$ is the inward vector of $F_3 - N_k$, and let q be the other point. Then near p , $[S_1] = [\alpha_k] \times [n]$ and near q , $[S_3] = [\alpha_k] \times (-[n])$. Therefore, at p ,

$$\begin{aligned} [S_1] \times [S_3] &= [\gamma_1] \times [v] \times [\alpha_k] \times [n] = \varepsilon [\beta_k] \times [v] \times [\alpha_k] \times [n] \\ &= \varepsilon [\alpha_k] \times [\beta_k] \times [v] \times [n] = \varepsilon [F_3] \times [v] \times [n] = \varepsilon [W] \end{aligned}$$

and the sign of p is ε . So the orientation of $\gamma^{(3,1)}$ is chosen in such a way that

$$[\gamma_3^{(3,1)} \cap F_3] = \varepsilon [\beta_k]$$

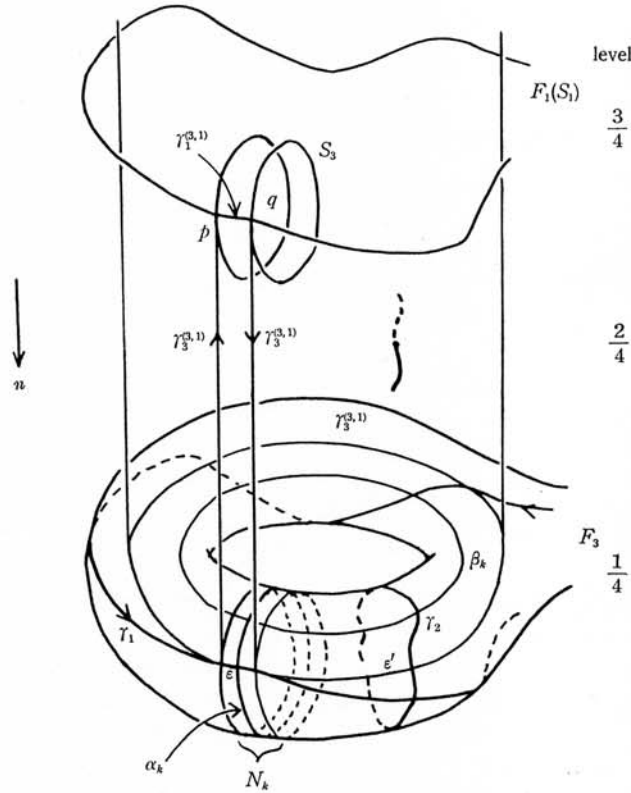


FIGURE 1

and that the orientation of the Whitney disk is

$$[\mathcal{A}^{(3,1)}] = [n] \times \varepsilon[\beta_k].$$

If there is an intersection point of γ_2 and $\gamma_3^{(3,1)}$ on F_3 , there is a corresponding intersection point of $\mathcal{A}^{(3,1)}$ and S_2 . Let the orientation of γ_2 be $\varepsilon'[\alpha_k]$ ($\varepsilon' = \pm 1$) near the point. Then the sign of the resulting intersection of $\mathcal{A}^{(3,1)}$ and S_2 can be computed as follows:

$$\begin{aligned} [\mathcal{A}^{(3,1)}] \times [S_2] &= [n] \times \varepsilon[\beta_k] \times [\gamma_2] \times [v] = [n] \times \varepsilon[\beta_k] \times \varepsilon'[\alpha_k] \times [v] \\ &= \varepsilon\varepsilon'[\alpha_k] \times [\beta_k] \times [v] \times [n] = \varepsilon\varepsilon'[W]. \end{aligned}$$

This Whitney disk $\mathcal{A}^{(3,1)}$ of type 1 may intersect S_2 outside the collar neighborhood of M , but the intersection occurs in pair and the algebraic sum is zero. Therefore

$$\sum_{\text{type 1}} \mathcal{A}^{(3,1)} \cdot S_2 = \sum \varepsilon \cdot \varepsilon' = \sum_{k=1}^t a_k^2 \cdot b_k^1.$$

Similarly

$$\sum_{\text{type 1}} \Delta^{(2,3)} \cdot S_1 = -\sum_{k=1}^t a_k^1 \cdot b_k^2$$

and

$$\sum_{\text{type 1}} \Delta^{(1,2)} \cdot S_3 = 0 .$$

(2) Whitney disks of type 2. In $W - M \times [0, 1)$, the intersection of S_i and S_j occurs as in Figure 2, corresponding to the intersection of the immersed disks d_i and d_j . We call the Whitney disks obtained from this intersection as the Whitney disks of type 2. Clearly, Whitney disks of type 2 occur in pairs and

$$\sum_{\text{type 2}} \Delta^{(i,j)} \cdot S_k = 0 .$$

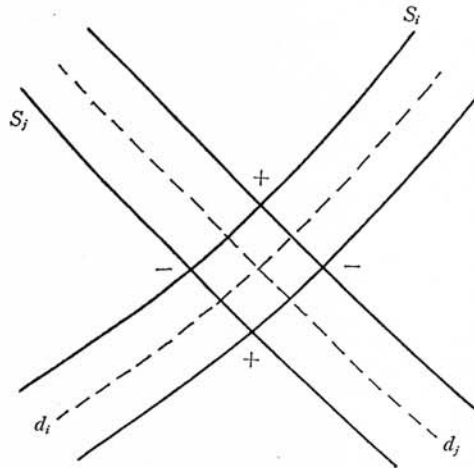


FIGURE 2

Combining (1) and (2), we obtain

$$\langle x_1, x_2, x_3 \rangle = \sum_{k=1}^t (a_k^2 \cdot b_k^1 - a_k^1 \cdot b_k^2)$$

by the definition of the Matsumoto tripling and Theorem is proved in case $I = (0)$.

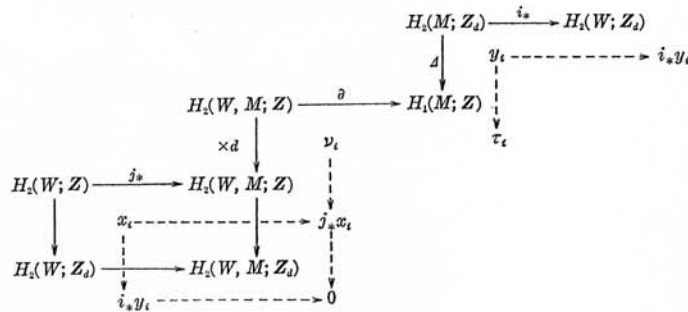
3. The proof of Theorem; part 2. Our plan of the proof is entirely the same as in the case $I = (0)$. However, an element of $H_2(M; \mathbb{Z}_d)$ cannot be realized by an embedded surface in general. This is the only point to study.

Now, we shall show how to construct a nice complex representing y_i . The Bockstein homomorphism $\Delta: H_2(M; \mathbb{Z}_d) \rightarrow H_1(M; \mathbb{Z})$ maps y_i to τ_i . Assume τ_i is non-trivial. For, otherwise, y_i can be realized by an embedded surface. Let l_i be a closed curve representing τ_i , and N be a tubular neighborhood of l_i . Then $H_1(\partial N; \mathbb{Z}) \approx \mathbb{Z} \oplus \mathbb{Z}$ is generated by a longitude ξ and a meridian η . Since $d\tau_i = 0$, the element $d \cdot \xi + e \cdot \eta$ is null-homologous in $M - \dot{N}$ for some integer e . Here, this number e is related to the linking number in the sense of Seifert [5] by $V(\tau_i, \tau_i) = e/d \pmod{1}$. Then the element $d \cdot \xi + e \cdot \eta$ is represented by a torus link of type (d, e) on ∂N . By the Pontrjagin-Thom construction, L_i bounds a properly embedded punctured surface T_i in $(M - \dot{N}, \partial N)$ (which is possibly non-connected). Let L_i^* be the fiberwise join of L_i and l_i . Then $T_i \cup L_i^*$ represents an element $y'_i \in H_2(M; \mathbb{Z}_d)$ and $\Delta(y'_i) = \tau_i$. This implies that $y_i - y'_i = z_i$ can be considered as an element of $H_2(M; \mathbb{Z})$. Moreover, the Mayer-Vietoris exact sequence shows that z_i is also an element of $H_2(M - \dot{N}; \mathbb{Z})$. Thus we can represent the homology class $y_i = y'_i + z_i \in H_2(M; \mathbb{Z}_d)$ again by the union of a properly embedded surface T'_i with $\partial T'_i = L_i$ and L_i^* .

The next lemma plays a key role in the proof.

LEMMA. $V(\tau_i, \tau_j) = 0$ for $i \neq j$.

PROOF. We shall consider the following diagram.



The element $\nu_i \in H_2(W, M; \mathbb{Z})$ is mapped to $j_* x_i$ by the homomorphism $\times d$, the multiplication by the number d . Then $\partial(\nu_i) = \tau_i$ by definition. Now, the linking number can be calculated in terms of an intersection number in the 4-manifold W . Namely,

$$V(\tau_i, \tau_j) = x_i \cdot \nu_j / d \pmod{1}.$$

Our assumption $x_i \cdot x_j = 0$ is valid with rational coefficients. Thus $V(\tau_i, \tau_j) = 0$ for $i \neq j$, completing the proof.

REMARK. This lemma states that for some 2-chain C_i such that $\partial C_i = d\tau_i$, the intersection number $C_i \cdot \tau_j = 0 \pmod{d}$. However, we can choose a nice loop l_j representing τ_j in such a way that $C_i \cdot l_j = 0$. Then for a 2-chain C_j with $\partial C_j = d \cdot l_j$, the intersection number $C_j \cdot l_i = 0$, where l_i is a closed curve defined by C_i .

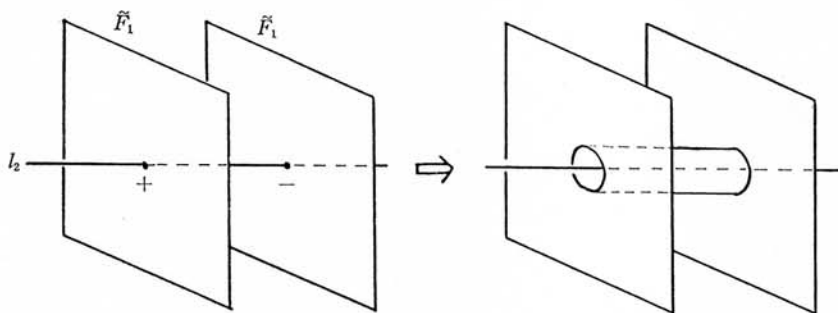


FIGURE 3

PROOF OF THEOREM IN CASE $I \neq (0)$. Take a complex \tilde{F}_1 which represents y_1 and contains a loop l_1 , representing τ_1 , as a singularity. Then there is a loop l_2 , representing τ_2 , with $\tilde{F}_1 \cdot l_2 = 0$ by the remark above. By performing surgery on \tilde{F}_1 as in Figure 3, if necessary, we may assume that $\tilde{F}_1 \cap l_2 = \emptyset$. The complex \tilde{F}_2 represents y_2 and contains a loop l_2 as a singularity. For the same reason as above, it may be assumed that $\tilde{F}_2 \cap l_1 = \emptyset$. The loop l_3 and the complex \tilde{F}_3 are similarly defined.

Now, we shall count the algebraic sum of the triple points $\tilde{F}_1, \tilde{F}_2, \tilde{F}_3$. By the construction of \tilde{F}_i , we see that $\tilde{F}_1 \cap \tilde{F}_3$ and $\tilde{F}_2 \cap \tilde{F}_3$ represent cycles on $\tilde{F}_3 - l_3$ in the sense of integral coefficients. Set them as

$$\sum_k (a_k^1[\alpha_k] + b_k^1[\beta_k]) + \sum_j c_j^1[\gamma_j]$$

$$\sum_k (a_k^2[\alpha_k] + b_k^2[\beta_k]) + \sum_j c_j^2[\gamma_j],$$

where $[\alpha_k], [\beta_k], [\gamma_j]$ is the canonical system of generators of $H_1(\tilde{F}_3 - l_3; \mathbf{Z})$ as in Figure 4. Then by the definition of the cup product,

$$((y_1^* \cup y_2^* \cup y_3^*) \cap [M]) = \sum (a_k^1 \cdot b_k^2 - a_k^2 \cdot b_k^1).$$

The next step of the proof is to perform surgery on \tilde{F}_i in W so that it is realized by an immersed 2-sphere. Take a properly immersed 2-disk D_i which represents $\nu_i \in H_2(W, M; \mathbf{Z})$ and satisfies $\partial D_i = l_i$. Then $F_i = \tilde{F}_i \cup d \cdot D_i$ is regarded as the image from a closed surface. And by

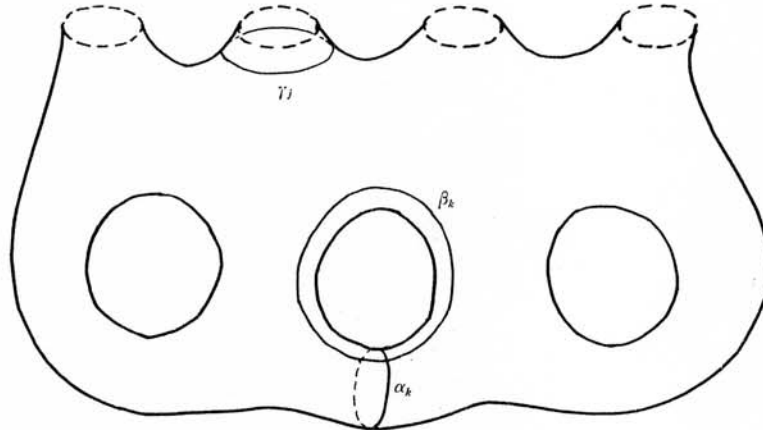


FIGURE 4

assumption, $D_i \cdot D_j = 0$ for $i \neq j$. The rest of the proof is entirely the same as before. Because in the computation of the Matsumoto tripling, the intersection number of S_k and Whitney disks $\Delta^{(i,j)}$'s, which lie in the complement of the collar of M in W , is zero modulo d for the same reason as in the case $d = 0$. This completes the proof of Theorem.

4. Some applications of Theorem. First of all we give

PROOF OF COROLLARY 1. Since W and W' are simply connected, $g_*[M] = [M']$ implies that f is of degree one, i.e., $f_*[W] = [W']$. Hence $f_*: H_2(W) \rightarrow H_2(W')$ is an epimorphism and the intersection ideal of $\{f_*x_1, f_*x_2, f_*x_3\}$ is contained in $I = I\{x_1, x_2, x_3\}$. Thus we have the reduction $\langle f_*x_1, f_*x_2, f_*x_3 \rangle'$ of $\langle f_*x_1, f_*x_2, f_*x_3 \rangle$ as an element of Z/I . We put $K^i(M; Z_d) = \text{Coker}(g_*: H^i(M'; Z_d) \rightarrow H^i(M; Z_d))$ and $K_i(M; Z_d) = \text{Ker}(g_*: H_i(M; Z_d) \rightarrow H_i(M'; Z_d))$. Since the map $g: M \rightarrow M'$ is of degree one, we have direct sum decompositions $H^i(M; Z_d) = H^i(M'; Z_d) \oplus K^i(M; Z_d)$ and $H_{3-i}(M; Z_d) = H_{3-i}(M'; Z_d) \oplus K_{3-i}(M; Z_d)$, which are compatible with the Poincaré duality isomorphism $\cap[M]$ and orthogonal with respect to the pairing $\langle, \rangle: H^i(M; Z_d) \otimes H^{3-i}(M; Z_d) \rightarrow Z_d$ given by $\langle \xi, \eta \rangle = (\xi \cup \eta) \cap [M]$, see [1, pp. 9-12].

For an element $x \in H_2(W; Z)$, let $y \in H_2(M; Z_d)$ be a mod d boundary reduction of x and let y^* be the Poincaré dual of y . If y' is a mod d boundary reduction of $x' = f_*x$, then we have $g_*y = y'$, since $i'_*: H_2(M'; Z_d) \rightarrow H_2(W'; Z_d)$ is monic. In order to prove Corollary 1, it suffices to show that

$$g^*(y_1'^* \cup y_2'^* \cup y_3'^*) \cap [M] = (y_1^* \cup y_2^* \cup y_3^*) \cap [M],$$

where $y_i^* \in H^1(M; \mathbf{Z}_d)$ and $y_i'^* \in H^1(M'; \mathbf{Z}_d)$ are the Poincaré duals of $y_i \in H_2(M; \mathbf{Z}_d)$ and $y_i' \in H_2(M'; \mathbf{Z}_d)$ such that $g_* y_i = y_i'$. Since the direct sum decompositions of $H_i(M)$ above are compatible with the Poincaré duality, we may write $g^* y_i'^* = y_i^* + u_i$ for some $u_i \in K^1(M)$. Moreover, they are orthogonal with respect to the pairing \langle, \rangle . It follows that

$$\begin{aligned} g^*(y_1'^* \cup y_2'^* \cup y_3'^*) \cap [M] \\ &= ((y_1^* + u_1) \cup (y_2^* + u_2) \cup (y_3^* + u_3)) \cap [M] \\ &= (y_1^* \cup y_2^* \cup y_3^*) \cap [M]. \end{aligned}$$

This completes the proof of Corollary 1.

Kaplan's number and Milnor's higher linking number. Let $L = \bigcup_{i=1}^3 k_i$ be a link in S^3 of three components k_1, k_2, k_3 with the mutual linking numbers zero. Regard S^3 as the boundary of a 4-ball B^4 and attach a 2-handle to S^3 by a 0-framing of each k_i . This gives us a compact simply connected 4-manifold W_L with boundary $M_L = \partial W_L$. Let $x_1, x_2, x_3 \in H_2(W_L; \mathbf{Z})$ be the homology classes of W_L corresponding to the cores of the attached 2-handles. Then there exist unique integral boundary reductions $y_i \in H_2(M_L; \mathbf{Z})$ of $x_i, i = 1, 2, 3$. Matsumoto [3] observed that $-\langle x_1, x_2, x_3 \rangle$ in W_L coincides with Milnor's higher linking number $\mu(L)$ of L , [4]. Meanwhile, Kaplan [2] has defined an invariant $T(L)$ and proved that $T(L) = (y_1^* \cup y_2^* \cup y_3^*) \cap [M_L]$. Thus we have

COROLLARY 2. *Let $L = \bigcup_{i=1}^3 k_i$ be a link as above. Then we have $T(L) = \mu(L)$.*

Normal singularity of complex algebraic surfaces. Let V be a complex algebraic surface in an open neighborhood of the origin O in a complex N -space C^N . Suppose that V contains O as a normal singularity. For sufficiently small $\varepsilon > 0$, a closed oriented 3-manifold $K_\varepsilon(O) = \{z \in V \mid |z| = \varepsilon\}$ is called a link of O in V . Sullivan [6] has proved that $(y_1^* \cup y_2^* \cup y_3^*) \cap [K_\varepsilon(O)] = 0$ for any elements $y_1, y_2, y_3 \in H_2(K_\varepsilon(O); \mathbf{Z})$. Hence we have

COROLLARY 3. *Let K be a link of a normal singularity in a complex algebraic surface. Then in a compact simply connected 4-manifold W with $\partial W = K$, three homology classes $x_1, x_2, x_3 \in H_2(W; \mathbf{Z})$ can be realized as mutually disjoint immersed 2-spheres in W if $j_* x_i = 0$ for $i = 1, 2, 3$, where j_* is induced by the inclusion map $j: W \rightarrow (W, M)$. In particular, if W is the so-called Milnor fiber of an isolated hypersurface singularity in C^3 with the monodromy $h_*: H_2(W, \mathbf{Z}) \rightarrow H_2(W, \mathbf{Z})$, then three invariant homology classes $x_1, x_2, x_3 \in H_2(W; \mathbf{Z})$ of h_* (i.e., $h_*(x_i) = x_i$) can be realized by mutually disjoint immersed 2-spheres.*

REMARK. We cannot expect the Z_d -analogue of Sullivan's result. In fact, consider the isolated hypersurface singularity; $z_1^d + z_2^d + z_3^d = 0$, ($d \geq 3$), whose link $K_c(\mathcal{O})$ is an S^1 -bundle of degree d over a surface of genus $(d-1)(d-2)/2$. Then there exist three elements $y_1, y_2, y_3 \in H_2(K_c(\mathcal{O}); \mathbb{Z}_d)$ such that $y_1^* \cup y_2^* \cup y_3^* \neq 0$.

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