3-MANIFOLDS AND 4-DIMENSIONAL SURGERY

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Abstract. Let X be a compact connected orientable Haken 3-manifold with boundary, and let $M(X)$ denote the 4-manifold $\partial(X \times D^2)$. We show that if $(f, b) : N \to M(X)$ is a degree 1 TOP normal map with trivial surgery obstruction in $L_4(\pi_1(M(X)))$, then (f, b) is TOP normally bordant to a homotopy equivalence $f' : N' \to M(X)$. Furthermore, for any CW-spine B of X, we have a UV^1-map $p : M(X) \to B$ and, for any $\epsilon > 0$, f' can be chosen to be a $p^{-1}(\epsilon)$-homotopy equivalence.

1. Introduction

Hegenbarth and Repovš [3] compared the controlled surgery exact sequence of Pedersen-Quinn-Ranicki [7] with the ordianry surgery sequence and observed the following:

Theorem 1 (Hegenbarth-Repovš). *Let M be a closed oriented TOP 4-manifold and $p : M \to B$ be a UV^1-map to a finite CW-complex such that the assembly map*

$$A : H_4(B; \mathbb{L}_\bullet) \to L_4(\pi_1(B))$$

is injective. Then the following holds: if $(f, b) : N \to M$ is a degree 1 TOP normal map with trivial surgery obstruction in $L_4(\pi_1(M))$, then (f, b) is TOP normally bordant to a $p^{-1}(\epsilon)$-homotopy equivalence $f' : N' \to M$ for any $\epsilon > 0$. In particular (f, b) is TOP normally bordant to a homotopy equivalence.

Remarks. (1) A map $f : N \to M$ is a $p^{-1}(\epsilon)$-*homotopy equivalence* if there is a map $g : M \to N$ and homotopies $H : g \circ f \simeq 1_N$ and $K : f \circ g \simeq 1_M$ such that all the arcs

$$[0,1] \xrightarrow{H(x,-)} N \xrightarrow{f} M \xrightarrow{p} B$$

$$[0,1] \xrightarrow{K(y,-)} M \xrightarrow{p} B$$

have diameter $< \epsilon$.
(2) $\mathbb{L}_\bullet$ is the 0-connective simply-connected surgery spectrum [8].
(3) The definition of UV^1-maps is given in the next section. We have an isomorphism $\pi_1(M) \cong \pi_1(B)$.
(4) This is true because the assembly map can be identified with the forget-control map $F : H_4(B; \mathbb{L}_\bullet) \to L_4(\pi_1(M))$ which sends the controlled surgery obstruction to the ordinary surgery obstruction. By the injectivity of this map, the vanishing of the ordinary surgery obstruction implies the vanishing of the controlled surgery obstruction.

For each torus knot K, Hegenbarth and Repovš [3] constructed a 4-manifold $M(K)$ and a UV^1-map $p : M(K) \to B$ to a CW-spine B of the exterior of K such that $A : H_4(B; \mathbb{L}_\bullet) \to L_4(\pi_1(B))$ is an isomorphism. The aim of this paper is to extend their construction as follows.

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Let X be a compact connected orientable 3-manifold with nonempty boundary. Then $M(X) = \partial(X \times D^2)$ is a closed orientable smooth 4-manifold with the same fundamental group as X. In fact, for any CW-spine B of X, one can construct a UV^1-map $p : M(X) \to B$.

Theorem 2. *If X is a compact connected orientable Haken 3-manifold with boundary, and B is any CW-spine of X, then there is a UV^1-map $p : M(X) \to B$, and the assembly map $A : H_4(B; \mathbb{L}_\bullet) \to L_4(\pi_1(B))$ is an isomorphism.*

Thus we can apply Theorem 1 to these 4-manifolds. Here is a list of such 3-manifolds X:

(1) the exterior of a knot or a non-split link [1],
(2) the exterior of an irreducible subcomplex of a triangulation of S^3 [9].

The author recently learned that Qayum Khan proved the following [4].

Theorem 3 (Khan). *Suppose M is a closed connected orientable PL 4-manifold with fundamental group π such that the assembly map*

$$A : H_4(\pi; \mathbb{L}_\bullet) \to L_4(\pi)$$

is injective, or more generally, the 2-dimensional component of its prime 2 localization

$$\kappa_2 : H_2(\pi; \mathbb{Z}_2) \to L_4(\pi)$$

is injective. Then any degree 1 normal map $(f, b) : N \to M$ with vanishing surgery obstruction in $L_4(\pi)$ is normally bordant to a homotopy equivalence $M \to M$.

In the examples constructed above, X's are aspherical; so Khan's theorem applies to the $M(X)$'s.

In §2, we give a general method to construct UV^m-maps, and finish the proof of Theorem 2 in §3.

2. Construction of UV^{m-1}-maps

A proper surjection $f : X \to Y$ is said to be UV^{m-1} if, for any $y \in Y$ and for any neighborhood U of $f^{-1}(y)$ in X, there exists a smaller neighborhood V of $f^{-1}(y)$ such that any map $K \to V$ from a complex of dimension $\leq m-1$ to V is homotopic to a constant map as a map $K \to U$. A UV^{m-1} map induces an isomorphism on π_i for $0 \leq i < m$ and an epimorphism on π_m. See [6, pp. 505–506] for the detail.

Let X be a connected compact n-dimensional manifold with nonempty boundary, and fix a positive integer m. We assume that X has a handlebody structure. Recall from [2, p.136] that X fails to have a handlebody structure if and only if X is an nonsmoothable 4-manifold.

Take the product $X \times D^m$ of X with an m-dimensional disk D^m, and consider its boundary $M(X) = \partial(X \times D^m)$, which is an $(n + m - 1)$-dimensional closed manifold.

Recall that a handlebody structure gives a CW-spine of X [5, p.107]. So, take any CW-spine B of X: there is a continuous map $q : \partial X \to B$ and X is homeomorphic to the mapping cylinder of q. The mapping cylinder structure extends q to a strong deformation retraction $\overline{q} : X \to B$. Define a continuous map $p : M(X) \to B$ to be the restriction of the composite map

$$X \times D^m \xrightarrow{\text{projection}} X \xrightarrow{\overline{q}} B$$

to the boundary.

Proposition 4. *For any CW-spine B of X, $p : M(X) \to B$ is a UV^{m-1}-map.*

Proof. First, let us set up some notations. $M(X)$ decomposes into two compact manifolds with boundary:

$$P = X \times S^{m-1} \ , \qquad Q = \partial X \times D^m \ .$$

For any subset S of B, define subsets $P_S \subset P$ and $Q_S \subset Q$ by

$$P_S = \overline{q}^{-1}(S) \times S^{m-1}, \quad Q_S = q^{-1}(S) \times D^m \ .$$

Then $p^{-1}(S) = P_S \cup Q_S$.

Let b be a point of B and take any open neighborhood U of $p^{-1}(b)$ in $M(X)$. Since $M(X)$ is compact, the map p is closed and hence there exists an open neighborhood $\widehat{U}$ of b in B such that $p^{-1}(\widehat{U}) \subset U$. Choose a smaller open neighborhood $\widehat{V} \subset \widehat{U}$ of b, such that the inclusion map $\widehat{V} \to \widehat{U}$ is homotopic to the constant map to b, and set $V = p^{-1}(\widehat{V})$.

Suppose that $\varphi : K \to V$ is a continuous map from an $(m-1)$-dimensional complex. We show that the composite map

$$\varphi' : K \xrightarrow{\varphi} V \xrightarrow{\text{inclusion map}} U$$

is homotopic to a constant map.

First of all, $Q_{\widehat{V}}$ has a core $q^{-1}(\widehat{V}) \times \{0\}$ of codimension m, and, by transversality, we may assume that $\varphi : K \to P_{\widehat{V}} \cup Q_{\widehat{V}}$ misses the core, and hence, we can homotop φ to a map into $P_{\widehat{V}}$. Since $P_{\widehat{V}}$ deforms into $\widehat{V} \times S^{m-1}$, we can further homotop φ to a map into $\widehat{V} \times S^{m-1}$. By the choice of $\widehat{V}$, φ' is homotopic to a map into $\{b\} \times S^{m-1}$. Pick any point $\overline{b} \in q^{-1}(b)$. Then this map is homotopic to a map

$$K \to \{\bar{b}\} \times S^{m-1} \subset \{\bar{b}\} \times D^m \subset Q_{\widehat{U}} \ .$$

Therefore φ' is homotopic to a constant map. □

Proposition 5. *If X has a handlebody structure, then $\pi_i(X) \cong \pi_i(M(X))$ for $i \leq m-1$, and $\pi_m(X)$ is a quotient of $\pi_m(M(X))$.*

Proof. This immediately follows from the proposition above, but we will give an alternative proof here.

Take any handle decomposition of X:

$$X = h_1 \cup h_2 \cup \cdots \cup h_N \ .$$

This defines the dual handle decomposition of X on ∂X, in which an n-handle of the original handlebody is a 0-handle. Since X is connected, one can cancel all the 0-handles of the dual handle decomposition. Thus we may assume that there are no n-handles in the handlebody structure of X.

The handlebody structure of X above gives rise to a handlebody structure of $X \times D^m$:

$$X \times D^m = h'_1 \cup h'_2 \cup \cdots \cup h'_N \ ,$$

where $h'_i = h_i \times D^m$ is a handle of the same index as h_i. So there are only 0-handles up to $(n-1)$-handles, and the dual handle decomposition of $X \times D^m$ on $M(X)$ has no handles of index $\leq m$. The result follows. □

3. Proof of Theorem 2

Roushon [10] proved the following (among other things):

Theorem 6 (Roushon). *Let X be a compact connected orientable Haken 3-manifold. Then the surgery structure set $\mathcal{S}(X \times D^n \text{ rel } \partial)$ is trivial for any $n \geq 2$.*

The vanishing of $\mathcal{S}(X \times D^n \ rel\ \partial)$ implies that the 4-periodic assembly maps [8]

$$A : H_i(X; \mathbb{L}_\bullet(\mathbb{Z})) \to L_i(\pi_1(X)) \qquad (i \in \mathbb{Z})$$

are all isomorphisms. Since

$$H_i(X; \mathbb{L}_\bullet(\mathbb{Z})) \cong H_i(X; \mathbb{L}_\bullet)$$

for $i \geq \dim B$, the 0-connective assembly map

$$A : H_4(X; \mathbb{L}_\bullet) \to L_4(\pi_1(X))$$

is also an isomorphism.

Let B be any CW-spine of X and let $p : M(X) \to B$ be the UV^1-map constructed in the previous section. Since B is a deformation retract of X, the assembly map

$$A : H_4(B; \mathbb{L}_\bullet) \to L_4(\pi_1(B))$$

is an isomorphism. This finishes the proof of Theorem 2.

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