Intelligence of Low Dimensional Topology 2006

Knots and 4-dimensional topological surgery

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M^n : an n-dim TOP manifold,

conn. ori. closed

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$$\{\text{surgery problems}\} \longrightarrow L_n(\pi)$$

$$(f : N^n \to M^n, b) \longmapsto \theta(f, b)$$

$\theta(f, b)$: surgery obstruction

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$\theta(f, b) = 0$

if can do surgery to get a htpy eq.

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e.g. 1, $\mathbb{Z}^n$, subexponential groups

[Freedman-Quinn, Krushkal-Q, . . .]

There are other results

that depend on <u>topology</u> of M.

- Krushkal-Lee (2002),

  π : free so probably not good

- Hegenbarth-Repovš (2006)

an example due to H-R

$K \subset S^3$: a knot

$$E(K) = S^3 - \mathring{N}(K)$$

$$M(K) = \partial(E(K) \times D^2)$$

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OK for $M(K)$, when K is a

torus knot.

Theorem

TOP surgery obstruction theory works for $M(K)$ for any knot K.

properties of $E(K)$ and $S^3 - K$

- homology S^1's
- aspherical
- $S^3 - K$ has a complete non-positively curved metric.

  [Leeb 1995]

properties of $M(K)$

- $\pi_1(M(K)) = \pi_1(E(K))$
- not aspherical

the idea of H-R

Construct a 2-dim spine B of $E(K)$ and a projection $q : E(K) \to B$, so that each $q^{-1}(x)$ is a wedge of intervals along one end.

Restrict the map

$$E(K) \times D^2 \xrightarrow{\text{proj.}} E(K) \xrightarrow{q} B$$

to ∂ and get the <u>control map</u>

$$p : M(K) \to B.$$

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$\Longrightarrow$ a controlled surgery exact sequence for p

[Pedersen-Quinn-Ranicki (2003)]

$\epsilon > \delta > 0$: sufficiently small

$\mathcal{N} = \{$surgery problems to $M(K)\}/\sim$

$\mathcal{S}(M(K)) = \{$htpy eq.'s to $M(K)\}/\sim$

$$\begin{array}{ccccc}
\mathcal{S}_{\epsilon,\delta}(M(K)) & \longrightarrow & \mathcal{N} & \longrightarrow & H_4(B;\mathbb{L}) \\
\downarrow & & \| & & \downarrow A \\
\mathcal{S}(M(K)) & \longrightarrow & \mathcal{N} & \longrightarrow & L_4(\pi)
\end{array}$$

The first row is exact [P-Q-R].

Want to show the second row is also exact.

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Claim: A is injective.

$$\begin{array}{ccc} H_4(B;\mathbb{L}) & \xrightarrow{\phi_*} & H_4(S^1;\mathbb{L}) \\ \Big\downarrow A & & \Big\downarrow A \\ L_4(\pi_1(B)) & \xrightarrow{\phi_*} & L_4(\pi_1(S^1)) \end{array}$$

$\phi : B \to S^1$: a homology equivalence

$$\begin{array}{ccc} H_4(B;\mathbb{L}) & \xrightarrow[\cong]{\phi_*} & H_4(S^1;\mathbb{L}) \\ \Big\downarrow A & & \Big\downarrow A \\ L_4(\pi_1(B)) & \xrightarrow{\phi_*} & L_4(\pi_1(S^1)) \end{array}$$

$\phi : B \to S^1$: a homology equivalence

$\Rightarrow$ top row is an isomorphism

$$\begin{array}{ccc} H_4(B;\mathbb{L}) & \xrightarrow[\cong]{\phi_*} & H_4(S^1;\mathbb{L}) \\ \Big\downarrow A & & \Big\downarrow A \\ L_4(\pi_1(B)) & \xrightarrow[\cong]{\phi_*} & L_4(\pi_1(S^1)) \end{array}$$

Bottom row is an isomorphism. [Arvinda-Farrell-Roushon, 1997]

This uses the metric on $S^3 - K \simeq B$.

$$\begin{array}{ccc} H_4(B;\mathbb{L}) & \xrightarrow[\cong]{\phi_*} & H_4(S^1;\mathbb{L}) \\ \Big\downarrow A & & \cong \Big\downarrow A \\ L_4(\pi_1(B)) & \xrightarrow[\cong]{\phi_*} & L_4(\pi_1(S^1)) \end{array}$$

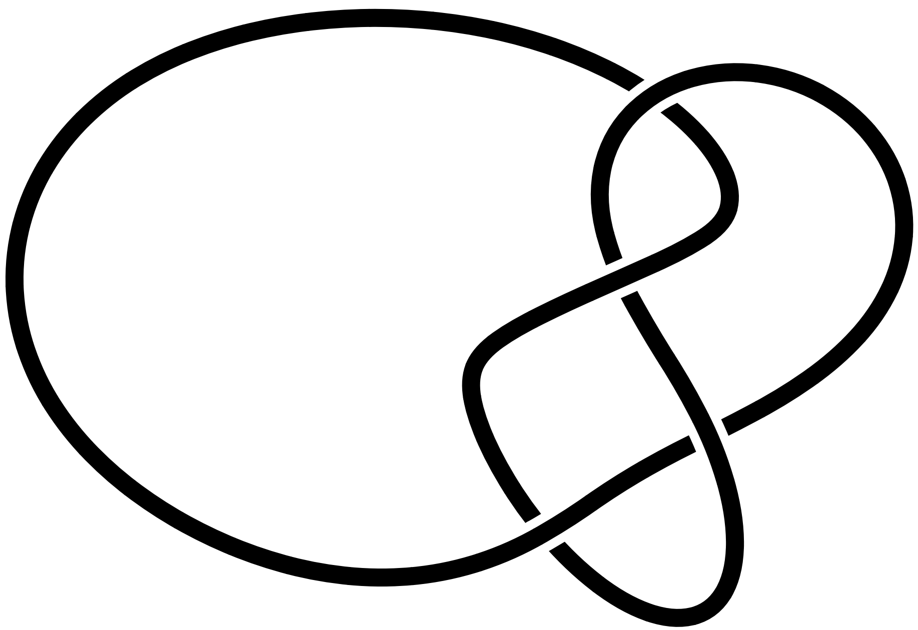
The assembly map A for S^1 is an isomorphism. [Browder, 1966]

$$\begin{array}{ccc} H_4(B;\mathbb{L}) & \xrightarrow[\cong]{\phi_*} & H_4(S^1;\mathbb{L}) \\ \downarrow{\scriptstyle A} & & \cong \downarrow{\scriptstyle A} \\ L_4(\pi_1(B)) & \xrightarrow[\cong]{\phi_*} & L_4(\pi_1(S^1)) \end{array}$$

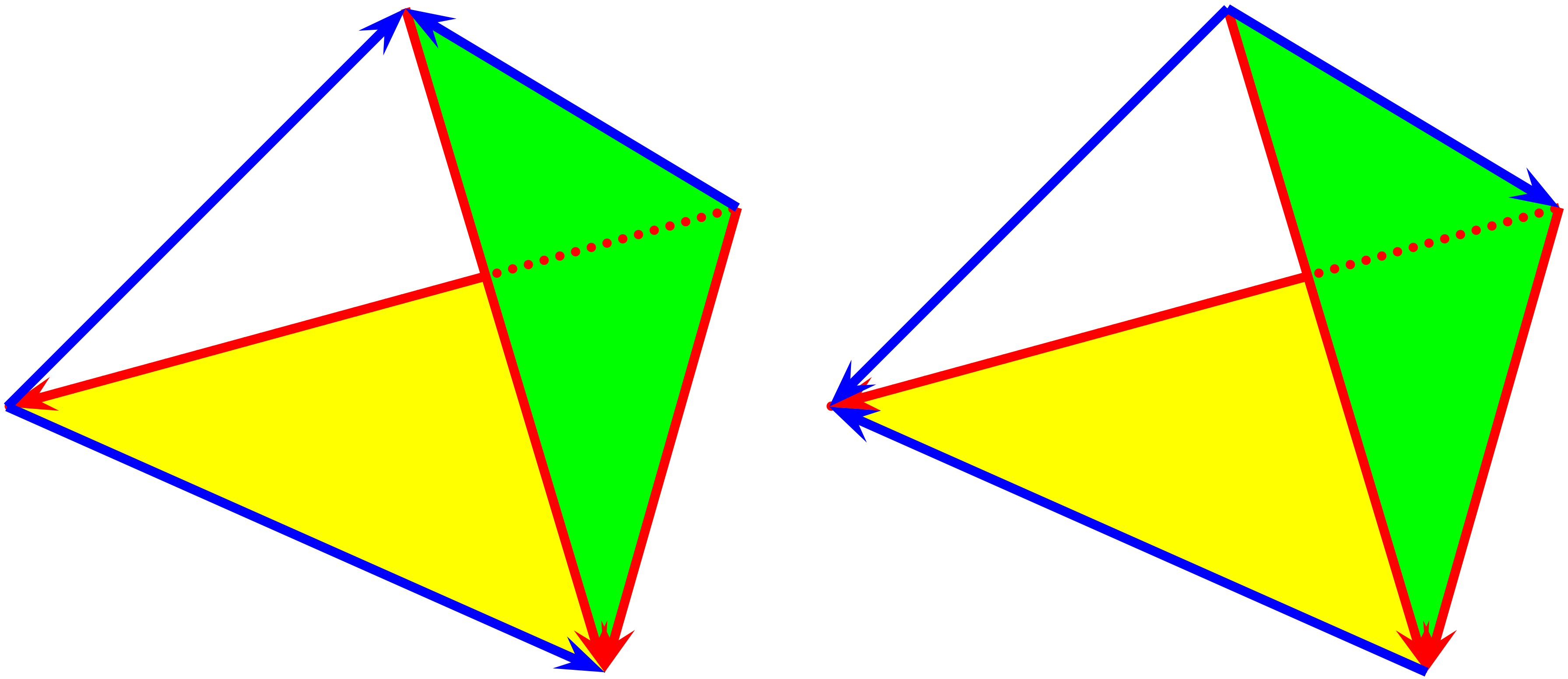
The assembly map A for B is also an isomorphism. $\Rightarrow$ exactness follows

Construction of the Spine B:

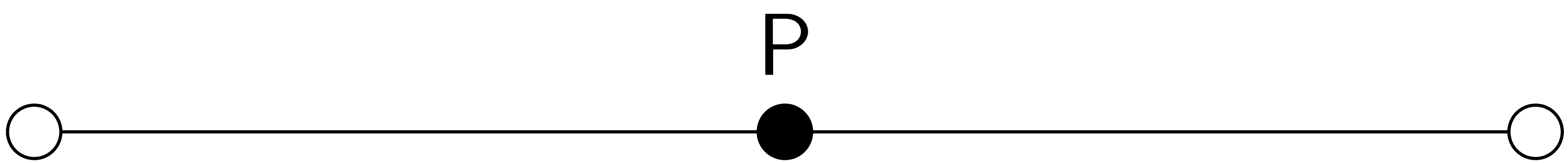
Figure Eight Knot Case



the ideal triangulation of the complement:



dual spine of an ideal 1-simplex



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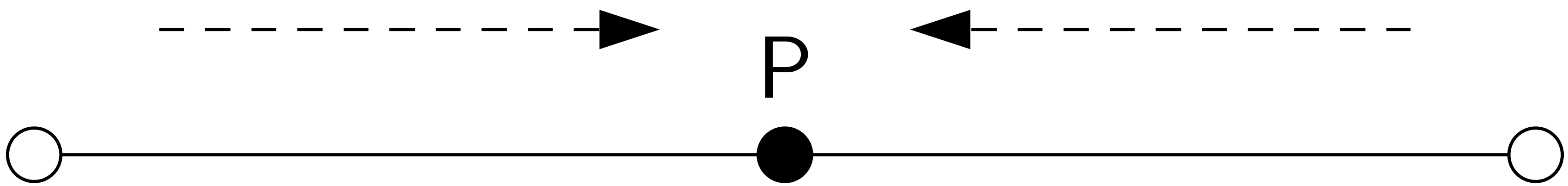
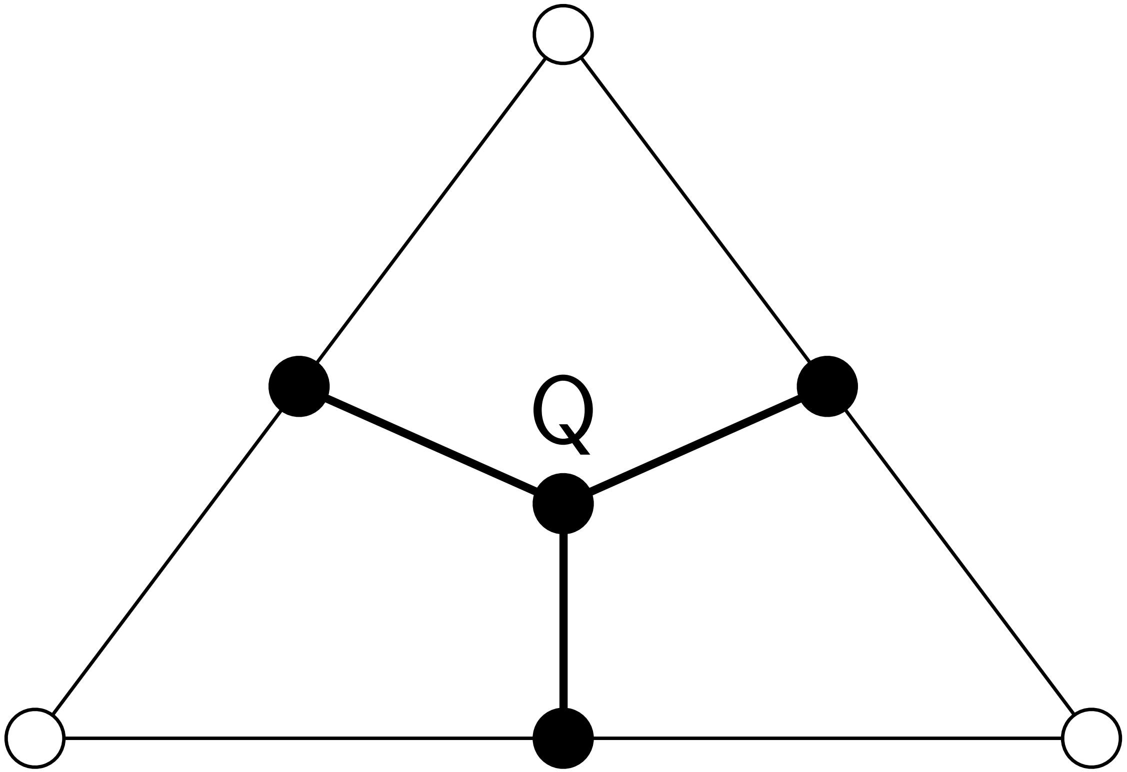
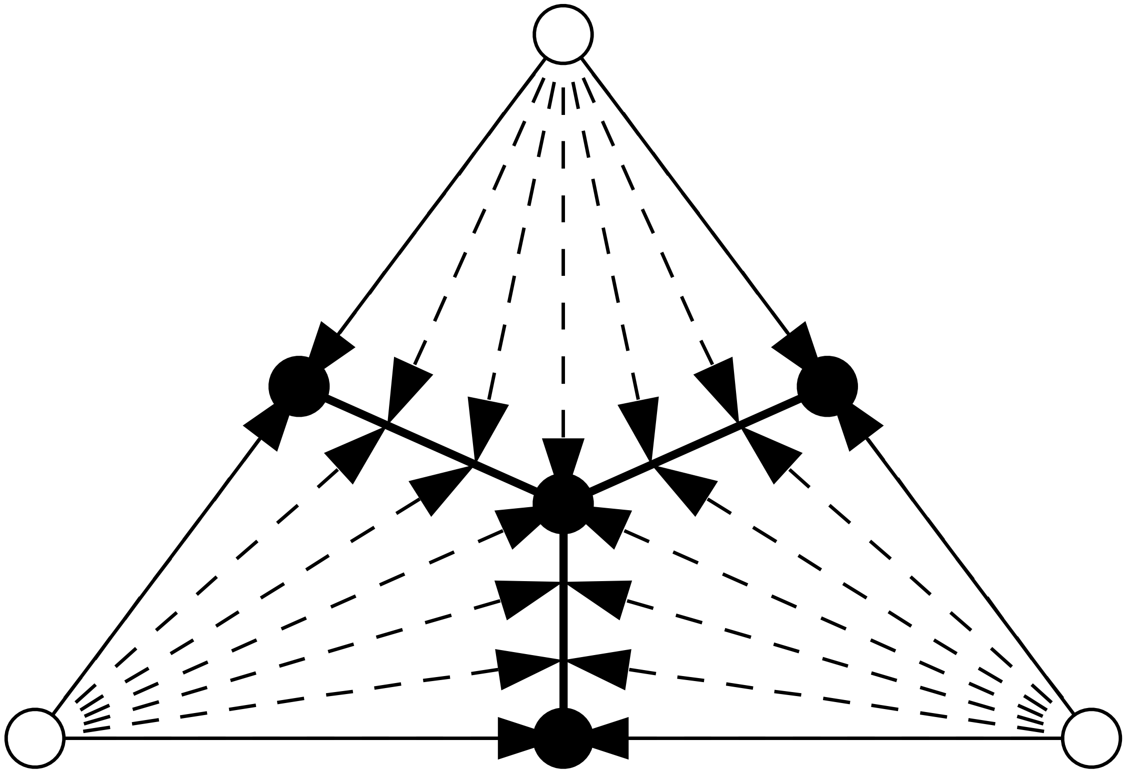


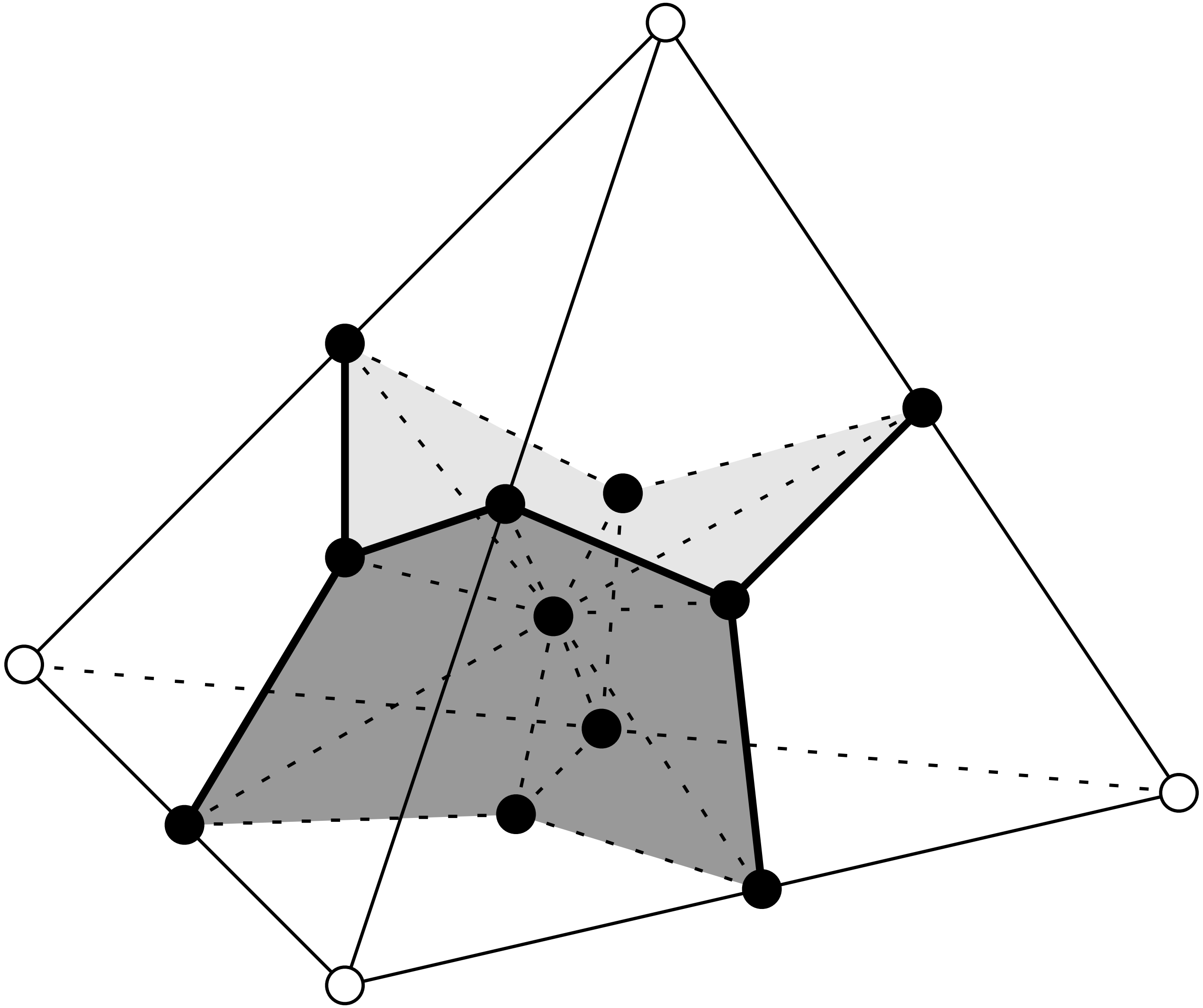
dual spine of an ideal 2-simplex



dual spine of an ideal 2-simplex

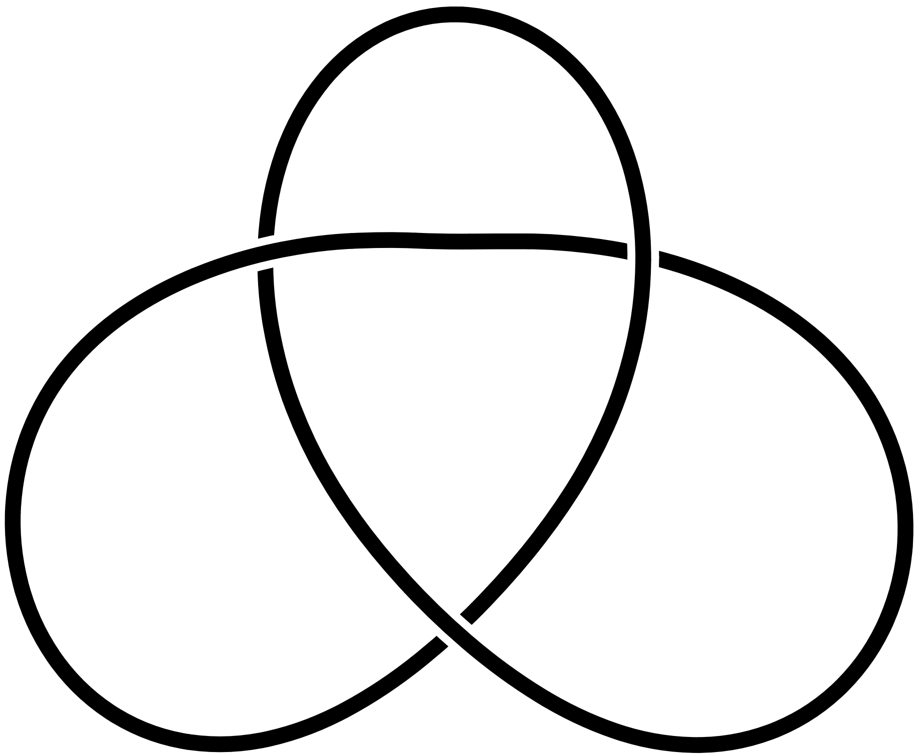


dual spine of an ideal 3-simplex

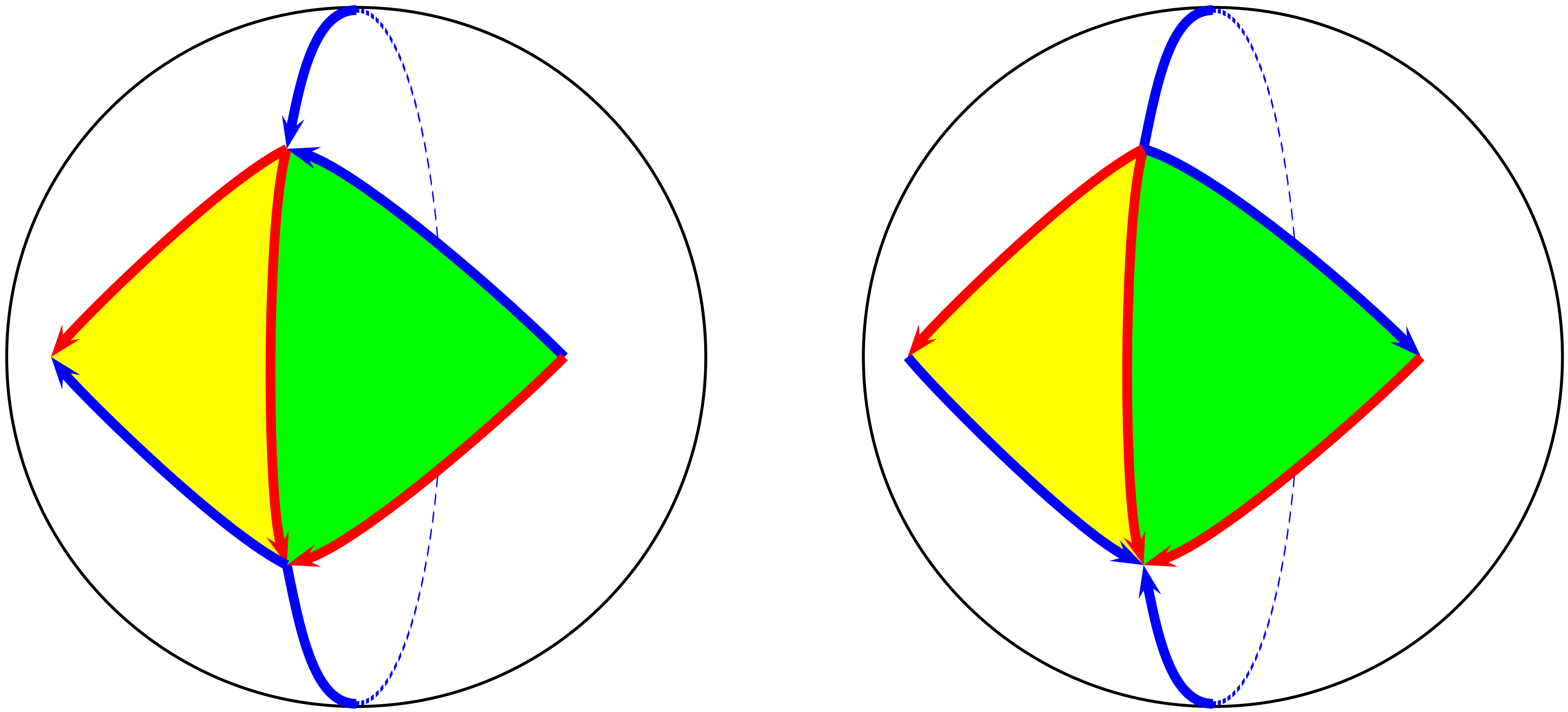


Construction of the Spine B:

Trefoil Knot Case



have a decomposition into ideal cells.



can similarly consider the dual spine.

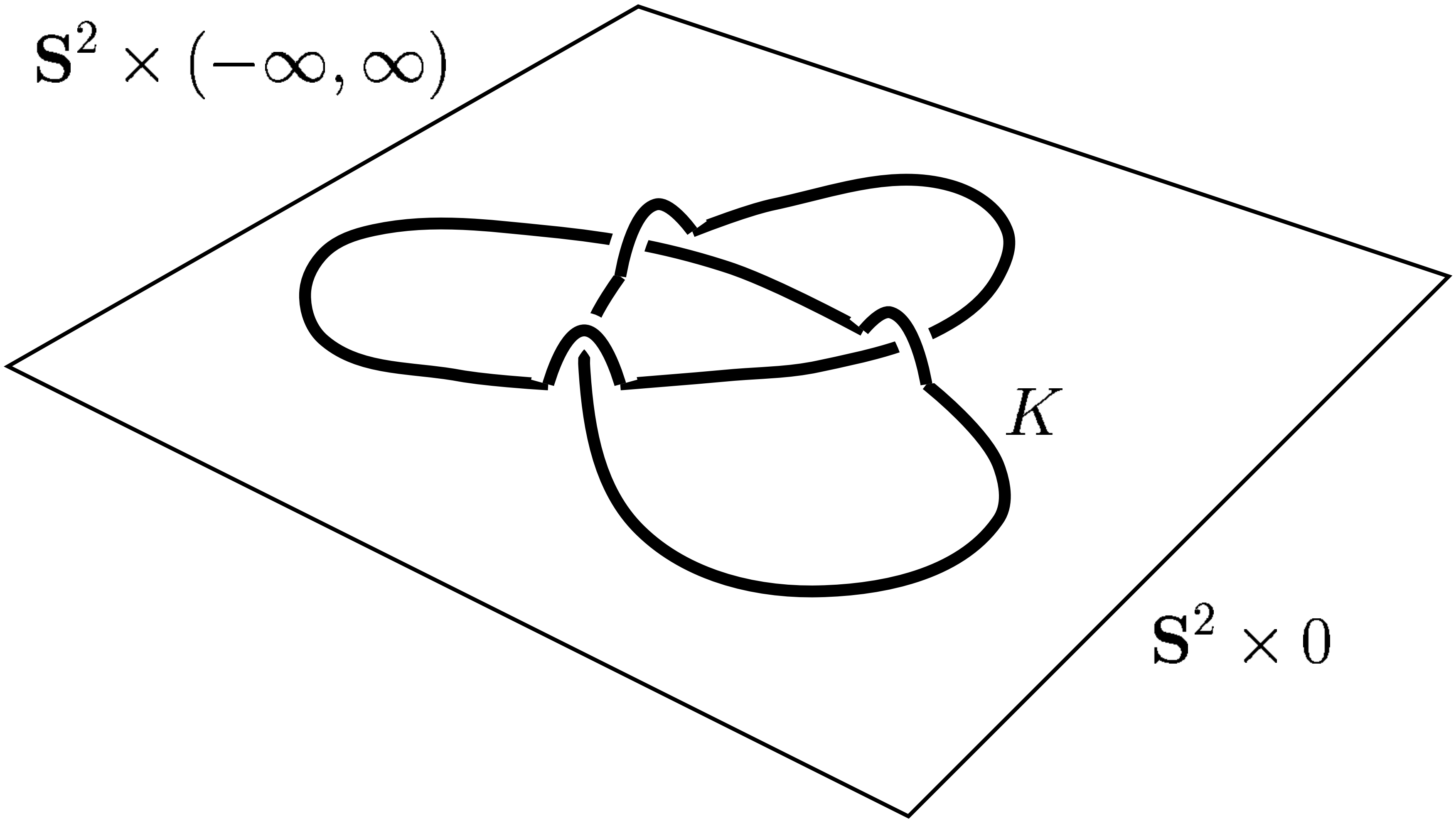
Construction of the Spine B: General Case

We use a simplified but weaker method of D. Thurston to construct a decomposition of the knot complement, and use its dual spine as B.

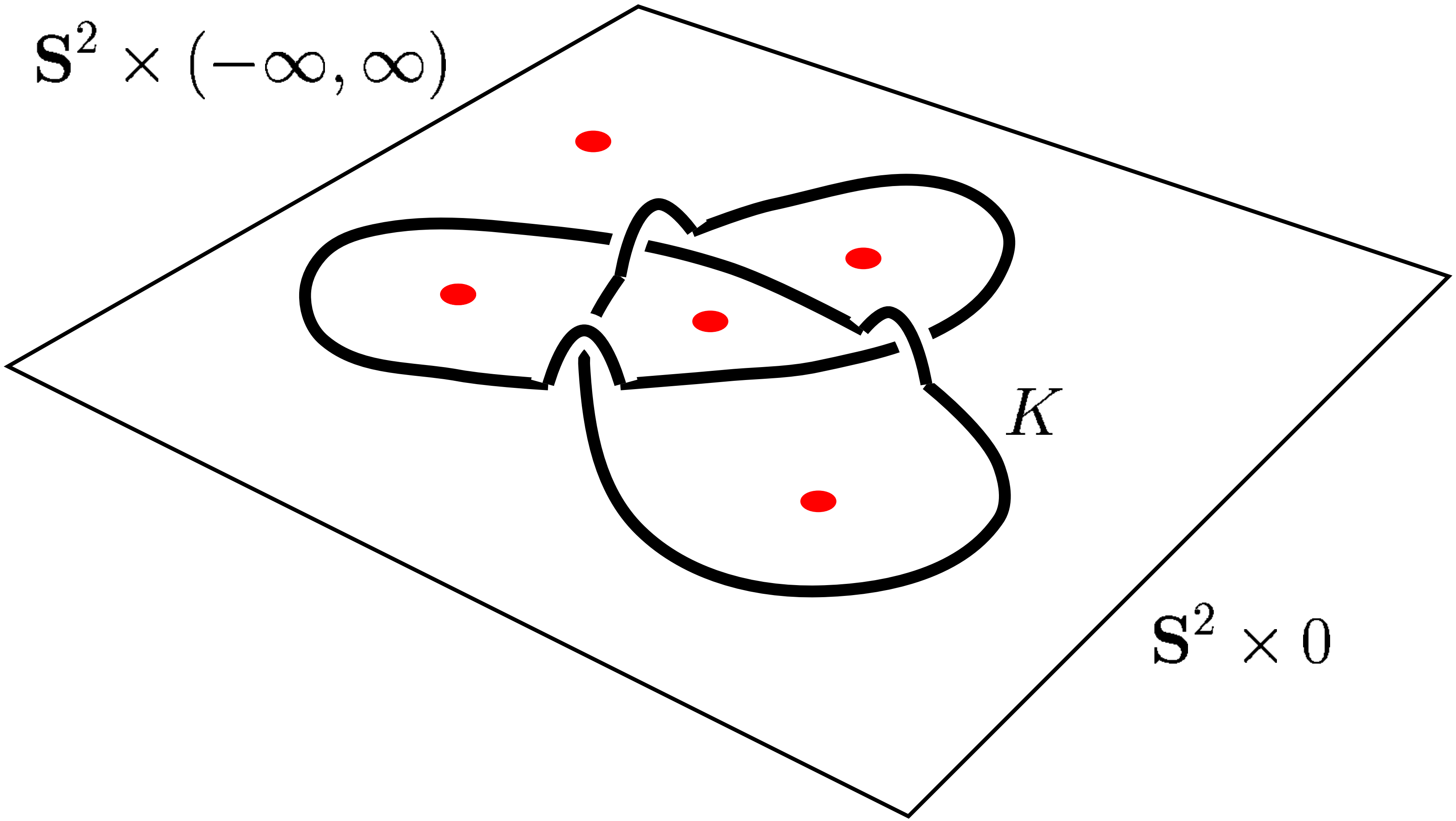
Identify S^3 with $S^2 \times (-\infty, \infty) \cup \{\pm\infty\}$,

and consider a knot projection to $S^2 \times 0$,

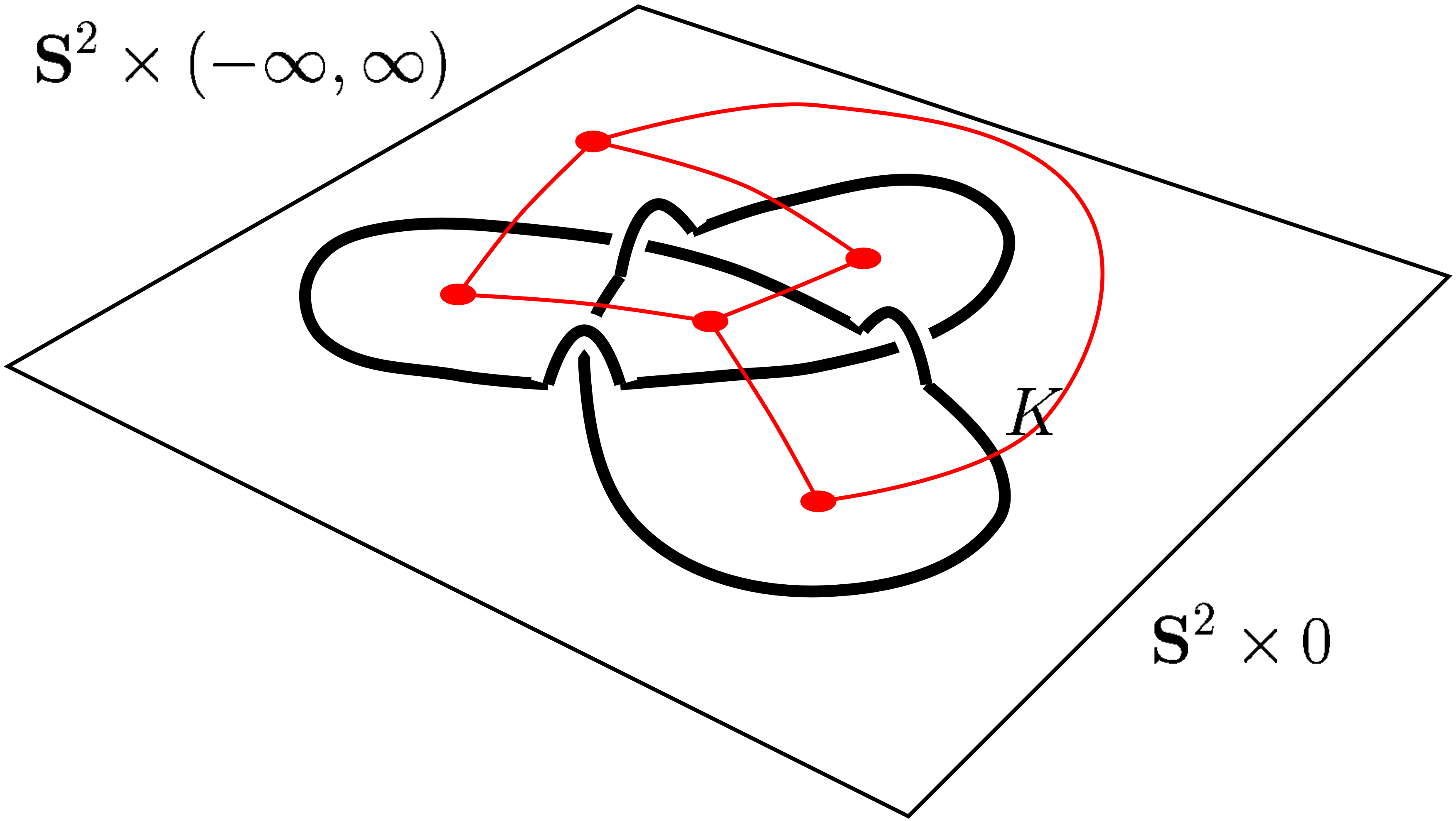
with n crossings.



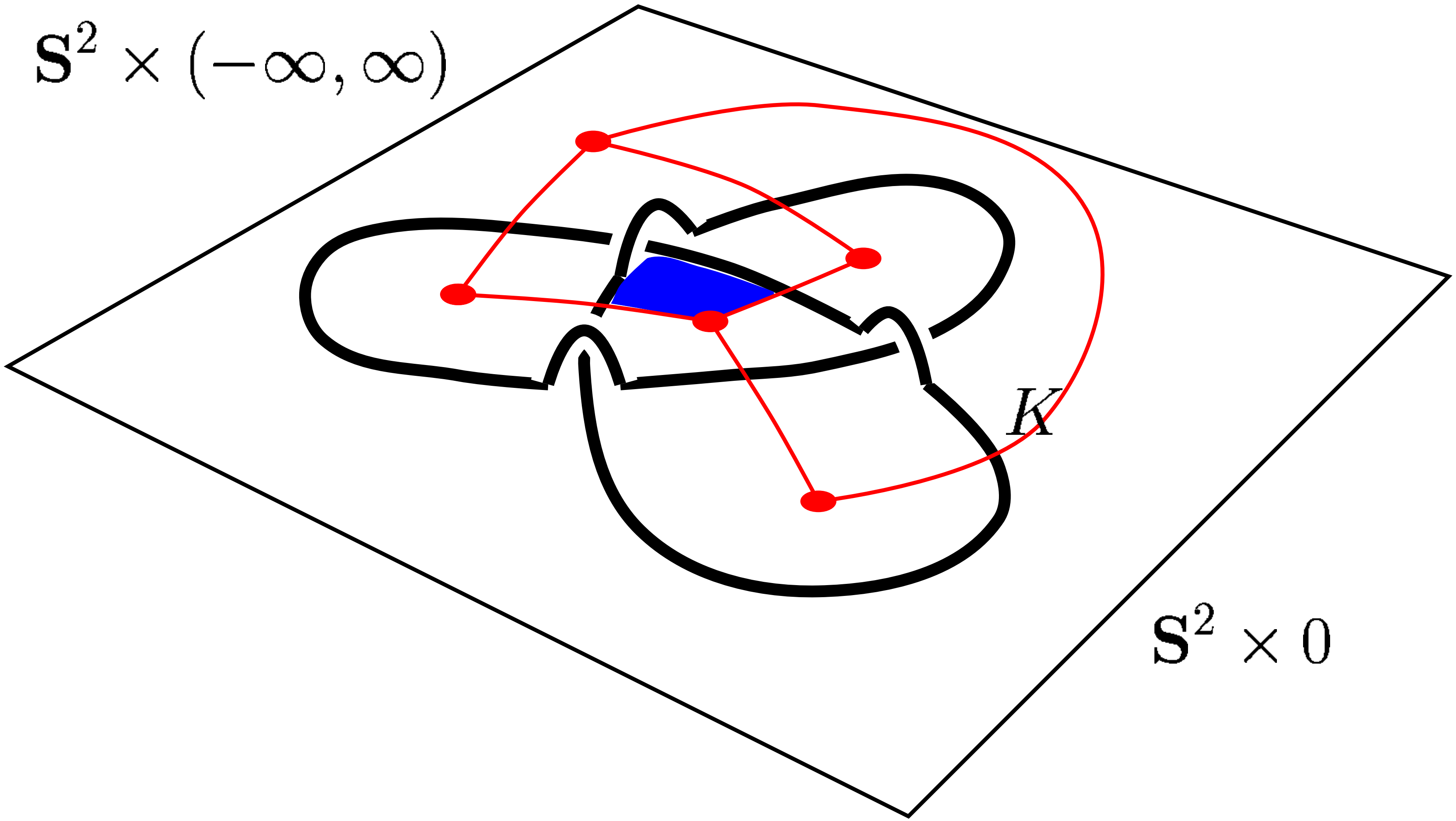
This divides $S^2 \times 0$ into several regions.



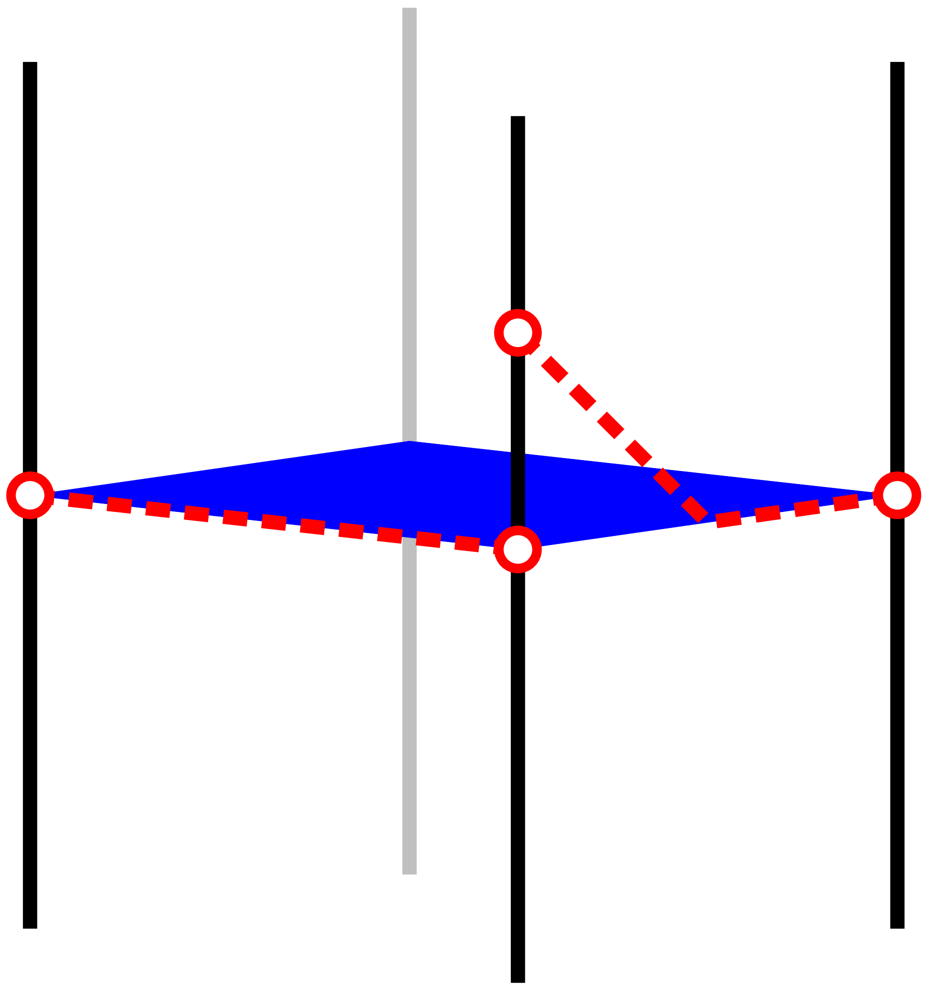
Pick a point from each region.



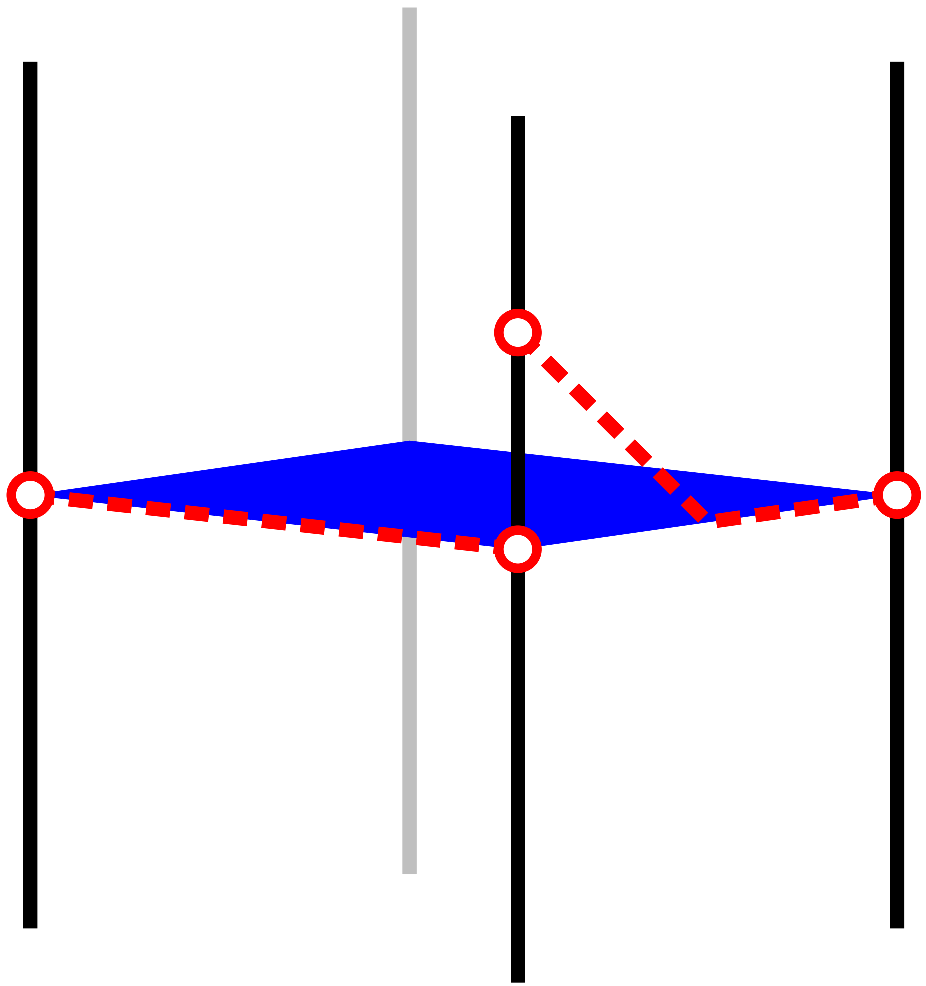
Connect the points as indicated above.



$S^2 \times 0$ decomposes into $4n$-many quadrangles R_i.

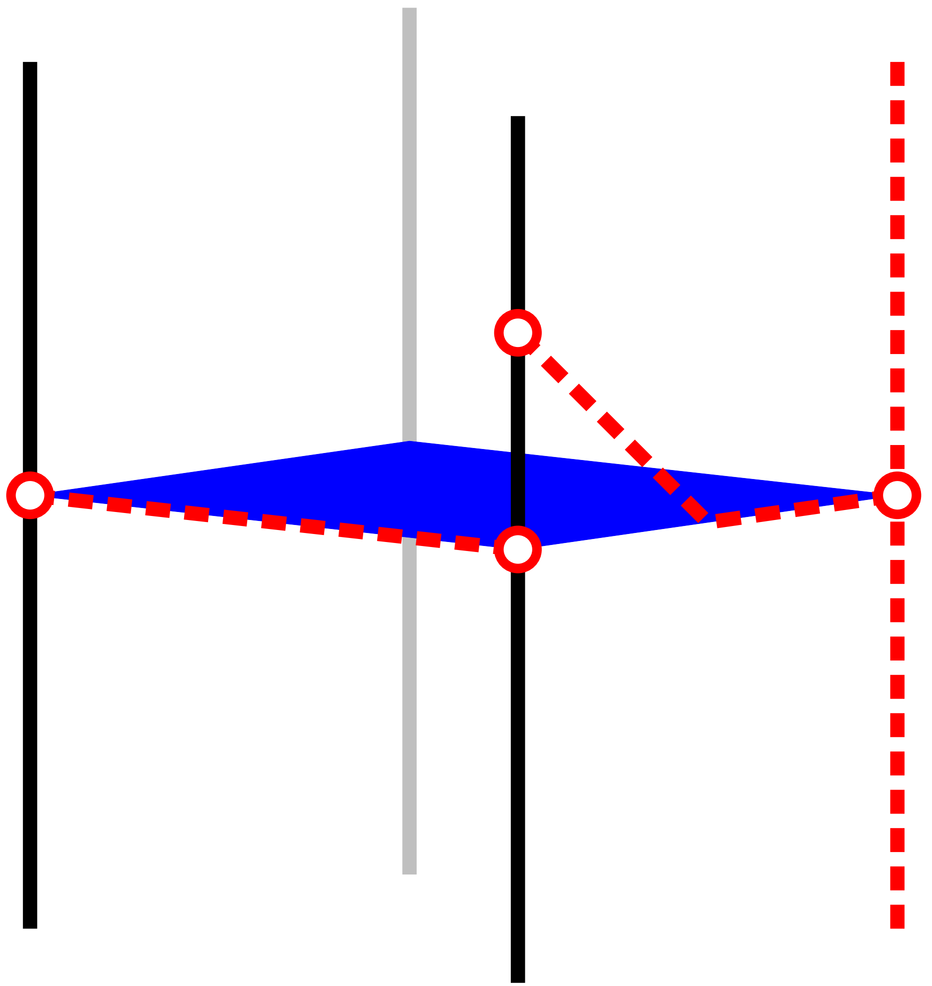


Roughly speaking $R_i \times (-\infty, \infty) - K$ are the desired cells.

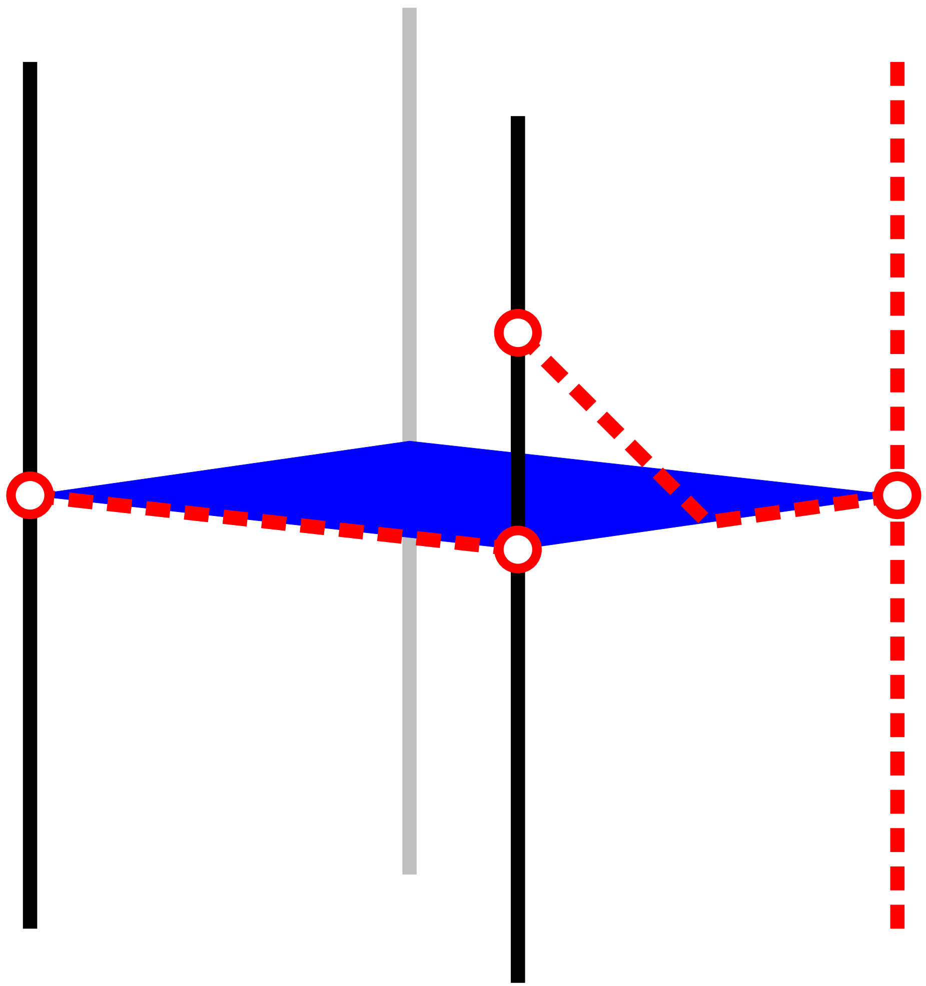


Unfortunately their union is not $S^3 - K$, but

$S^3 - \{\pm\infty\} - K$.



So pick a point on K and dig tunnels to $\pm\infty$. This affects four cells.



This gives a decomposition into ideal cells.

Now use the dual spine.

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