1. Preliminaries.

In this section we review the basics on chain complexes and chain maps. Chain complexes in the category of finitely generated free or projective modules over some ring R are the most important, but we also need to handle other chain complexes. We fix a category $\mathbb{A}$ and consider the chain complexes made up of objects and morphisms in the category. We require that the category $\mathbb{A}$ be *additive*; *i.e.* $\mathbb{A}$ satisfies the following four conditions:

1. The set $\operatorname{Hom}(A, B)$ of the morphisms from A to B is an abelian group for each A, $B \in \operatorname{Obj}\mathbb{A}$.
2. Composition of morphisms is bilinear.
3. $\mathbb{A}$ is equipped with a distinguished object 0 such that $\operatorname{Hom}(A, 0) = 0$, $\operatorname{Hom}(0, A) = 0$ for each $A \in \operatorname{Obj}\mathbb{A}$.
4. For each A, $B \in \operatorname{Obj}\mathbb{A}$, there is an object $A \oplus B$ together with two morphisms

$$A \xleftarrow{p_A} A \oplus B \xrightarrow{p_B} B$$

such that, for any $X \in \operatorname{Obj}\mathbb{A}$ and for any pair of morphisms $f_A \in \operatorname{Hom}(X, A)$, $f_B \in \operatorname{Hom}(X, B)$, there exists a unique morphism $f : X \to A \oplus B$ which makes the following diagram commute.

$$\begin{tikzcd} & X \arrow[dl, "f_A"'] \arrow[d, dotted, "\exists!\ f"] \arrow[dr, "f_B"] & \\ A & A \oplus B \arrow[l, "p_A"] \arrow[r, "p_B"'] & B \end{tikzcd}$$

Under the conditions 1 – 3, the condition 4 is equivalent to each of the following:

4′. For each A, $B \in \operatorname{Obj}\mathbb{A}$, there is an object $A \oplus B$ together with two morphisms

$$A \xrightarrow{i_A} A \oplus B \xleftarrow{i_B} B$$

such that, for any $Y \in \operatorname{Obj}\mathbb{A}$ and for any pair of morphisms $g_A \in \operatorname{Hom}(A, Y)$, $g_B \in \operatorname{Hom}(B, Y)$, there exists a unique morphism $g : A \oplus B \to Y$ which makes the following diagram commute:

$$\begin{tikzcd} & Y & \\ A \arrow[ur, "g_A"] \arrow[r, "i_A"'] & A \oplus B \arrow[u, dotted, "g"] & B \arrow[ul, "g_B"'] \arrow[l, "i_B"] \end{tikzcd}$$

4″. For each A, $B \in \operatorname{Obj}\mathbb{A}$, there is an object $A \oplus B$ together with four morphisms

$$\begin{tikzcd} A \arrow[r, shift left, "i_A"] & A \oplus B \arrow[l, shift left, "p_A"] \arrow[r, shift left, "p_B"] & B \arrow[l, shift left, "i_B"] \end{tikzcd}$$

such that

$$p_A i_A = 1_A, \quad p_B i_A = 0, \quad p_B i_B = 1_B, \quad p_A i_B = 0,$$
$$i_A p_A + i_B p_B = 1_{A \oplus B} \ .$$

Notation. The morphisms $f : X \to A \oplus B$ and $g : A \oplus B \to Y$ above will be denoted $\begin{pmatrix} f_A \\ f_B \end{pmatrix}$ and $(\, g_A \quad g_B \,)$ in the matrix form. More generally a morphism $f : A_1 \oplus \ldots \oplus A_m \to B_1 \oplus \ldots \oplus B_n$ will be expressed as a matrix

$$\begin{array}{c c} & \begin{array}{ccc} A_1 & \ldots & A_m \end{array} \\ \begin{array}{c} B_1 \\ \vdots \\ B_n \end{array} & \begin{pmatrix} f_{11} & \ldots & f_{1m} \\ \vdots & \ddots & \vdots \\ f_{n1} & \ldots & f_{nm} \end{pmatrix} \end{array}$$

for some morphisms $f_{ij} : A_j \to B_i$.

Definitions.

(1) A *chain complex* C *in* $\mathbb{A}$ is a sequence of morphisms in $\mathbb{A}$

$$C \ : \ \ldots \xrightarrow{d_C} C_{r+1} \xrightarrow{d_C} C_r \xrightarrow{d_C} C_{r-1} \xrightarrow{d_C} \ldots$$

such that $d_C^2 = 0 : C_{r+1} \to C_{r-1}$ for every $r \in \mathbb{Z}$. It is *n-dimensional* if $C_r = 0$ for every $r > n$ and for every $n < 0$.

(2) A *chain map* $f : C \to D$ between two chain complexes in $\mathbb{A}$ is a collection

$$\{f_r : C_r \to D_r\}$$

of morphisms such that

$$d_D f_r = f_{r-1} d_C : C_r \to D_{r-1}$$

for every r.

(3) A *chain homotopy* $h : f \simeq f' : C \to D$ between two chain maps $f, f' : C \to D$ is a collection

$$\{h_r : C_r \to D_{r+1}\}$$

of morphisms such that

$$d_D h + h d_C \ = \ f' - f \ : \ C_r \ \to \ D_r \ .$$

(4) A chain map $f : C \to D$ is an *isomorphism* if there is a chain map $g : D \to C$ such that

$$gf = 1 : C \to C, \qquad fg = 1 : D \to D\ .$$

A chain map $f : C \to D$ is a *chain equivalence* if there is a chain map $g : D \to C$ such that

$$gf \simeq 1 : C \to C, \qquad fg \simeq 1 : D \to D\ .$$

(5) The *algebraic mapping cone* $\mathcal{C}(f)$ of a chain map $f : C \to D$ is defined by

$$d_{\mathcal{C}(f)} \;=\; \begin{pmatrix} d_D & (-)^{r-1}f \\ 0 & d_C \end{pmatrix} \;:\; \mathcal{C}(f)_r \;=\; D_r \oplus C_{r-1} \to \mathcal{C}(f)_{r-1} \;=\; D_{r-1} \oplus C_{r-2}.$$

(6) The *suspension* SC and the *desuspension* ΩC of a chain complex C are defined by:

$$\begin{aligned} (d_{SC})_r \;&=\; (d_C)_{r-1} \;:\; (SC)_r \;=\; C_{r-1} \longrightarrow C_{r-2} \;=\; (SC)_{r-1} \\ (d_{\Omega C})_r \;&=\; (d_C)_{r+1} \;:\; (\Omega C)_r \;=\; C_{r+1} \longrightarrow C_r \;=\; (\Omega C)_{r-1}. \end{aligned}$$

For a chain map $f : C \to D$, its *suspension* $Sf : SC \to SD$ and *desuspension* $\Omega f : \Omega C \to \Omega D$ are defined by

$$\begin{aligned} (Sf)_r \;&=\; -f_{r-1} \;:\; (SC)_r \;=\; C_{r-1} \longrightarrow D_{r-1} \;=\; (SD)_r \\ (\Omega f)_r \;&=\; -f_{r+1} \;:\; (\Omega C)_r \;=\; C_{r+1} \longrightarrow D_{r+1} \;=\; (\Omega D)_r\ . \end{aligned}$$

The minus sign is used so that we have identities

$$\mathcal{C}(Sf) \;=\; S(\mathcal{C}f) \quad \text{and} \quad \mathcal{C}(\Omega f) \;=\; \Omega\mathcal{C}(f)\ .$$

But note that these behave badly with respect to compositions:

$$S(gf) = -(Sg)(Sf) \quad \text{and} \quad \Omega(gf) = -(\Omega g)(\Omega f)\ .$$

(7) A chain complex B is a *subcomplex* of a chain complex C if $C_r = B_r \oplus B'_r$ for some $B'_r \in \mathrm{Obj}\,\mathbb{A}$, and the composition

$$B_r \xrightarrow{i_B} B_r \oplus B'_r \xrightarrow{d_C} B_{r-1} \oplus B'_{r-1} \xrightarrow{p_{B'}} B'_{r-1}$$

is equal to 0; in ohter words, C is the algebraic mapping cone of a chain map $f : \Omega B' \to B$ for some chain complex B'. The chain complex B' is called the *quotient* and is denoted C/B. The morphisms i_B define a chain map $i : B \to C$ called the *inclusion map*, and the morphisms $p_{B'}$ define a chain map $p : C \to C/B$ called the *projection map.*

(8) If $C = \{(C_r, d_r)\}$ is a chain complex and ϵ is an integer, then $\epsilon \bullet C$ will denote the chain complex $\{C_r, \epsilon d_r\}$. The equality

$$\epsilon \bullet \mathcal{C}(f : C \to D) = \mathcal{C}(\epsilon f : \epsilon \bullet C \to \epsilon \bullet D)$$

can be easily verified. When $\epsilon = \pm 1$, $\epsilon \bullet C$ is isomorphic to C; an isomorphism is given by $\epsilon^r 1_{C_r} : C_r \to C_r$.

(9) Given chain complexes C and D in $\mathbb{A}$, let $\mathrm{Hom}_{\mathbb{A}}(C, D)$ be the $\mathbb{Z}$-module chain complex defined by

$$\begin{aligned} d_{\mathrm{Hom}_{\mathbb{A}}(C,D)} : \mathrm{Hom}_{\mathbb{A}}(C, D)_n = \sum_{q-p=n} \mathrm{Hom}_{\mathbb{A}}(C_p, D_q) \to \mathrm{Hom}_{\mathbb{A}}(C, D)_{n-1}; \\ f \mapsto d_D f + (-)^q f d_C \quad (: C_{q-n+1} \to C_q). \end{aligned}$$

(10) Given a right R-module chain complex C and a left R-module chain complex D, let $C \otimes_R D$ be the $\mathbb{Z}$-module chain complex defined by

$$\begin{aligned} d_{C\otimes_R D} : (C \otimes_R D)_n = \sum_{p+q=n} C_p \otimes_R D_q \to (C \otimes_R D)_{n-1}; \\ x \otimes y \mapsto x \otimes d_D(y) + (-)^q d_C(x) \otimes y \ . \end{aligned}$$

(11) When C,C', D, D' are left R-module chain complexes, there is a $\mathbb{Z}$-module chain map $\tau : \mathrm{Hom}_R(C, D) \otimes_{\mathbb{Z}} \mathrm{Hom}_R(C', D') \to \mathrm{Hom}_R(C \otimes_{\mathbb{Z}} C', D \otimes_{\mathbb{Z}} D')$ defined by

$$\begin{aligned} \tau : \mathrm{Hom}_R(C, D)_m \otimes_{\mathbb{Z}} \mathrm{Hom}_R(C', D')_n \to \mathrm{Hom}_R(C \otimes_{\mathbb{Z}} C', D \otimes_{\mathbb{Z}} D')_{m+n} \\ f \otimes g \mapsto \{x \otimes y (\in C_r \otimes_{\mathbb{Z}} C'_s) \mapsto (-)^{(m-r)s} f(x) \otimes g(y)\}. \end{aligned}$$

Next we review some properties of the algebraic mapping cone.

Proposition. *A chain map $f : C \to D$ is a chain equivalence if and only if its algebraic mapping cone $\mathcal{C}(f)$ is chain contractible.*

Proof: Suppose f is a chain equivalence. By definition, there exist a chain map $g : D \to C$ and chain homotopies

$$h : gf \simeq 1 : C \to C, \qquad k : fg \simeq 1 : D \to D \ .$$

Then the following matrix defines a chain contraction of $\mathcal{C}(f)$:

$$\begin{pmatrix} k + (fh - kf)g & (-)^r(fh - kf)h \\ (-)^r g & h \end{pmatrix}$$
$$: \mathcal{C}(f)_r = D_r \oplus C_{r-1} \to D_{r+1} \oplus C_r = \mathcal{C}(f)_{r+1}.$$

Conversely suppose a chain contraction $\Gamma : 0 \simeq 1 : \mathcal{C}(f) \to \mathcal{C}(f)$ is given. Define $g_r : D_r \to C_r$, $h_{r-1} : C_{r-1} \to C_r$, and $k_r : D_r \to D_{r+1}$ by:

$$\Gamma = \begin{pmatrix} k & * \\ (-)^r g & h \end{pmatrix} : \mathcal{C}(f)_r = D_r \oplus C_{r-1} \to D_{r+1} \oplus C_r = \mathcal{C}(f)_{r+1} \ .$$

Then $g : D \to C$ is a chain map and the equalities $dh + hd = 1 - gf$, $dk + kd = 1 - fg$ hold. □

Definition. A *triad* Γ of chain complexes

$$\Gamma : \begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, squiggly, "h"] & D \arrow[d, "g'"] \\ C' \arrow[r, "f'"'] & D' \end{tikzcd}$$

consists of chain maps

$$f : C \to D, \quad f' : C' \to D', \quad g : C \to C', \quad g' : D \to D'$$

and a chain homotopy $h : g'f \simeq f'g : C \to D'$.

Such a triad induces a chain map $(g, g'; h) : \mathcal{C}(f) \to \mathcal{C}(f')$:

$$(g, g'; h) = \begin{pmatrix} g' & (-)^r h \\ 0 & g \end{pmatrix} : \mathcal{C}(f)_r = D_r \oplus C_{r-1} \to \mathcal{C}(f')_r = D'_r \oplus C'_{r-1}$$

Conversely, if two chain maps $f : C \to D$, $f' : C' \to D'$ and a chain map $G : \mathcal{C}(f) \to \mathcal{C}(f')$ are given, and if G can be written using a matrix of the form

$$\begin{pmatrix} g' & \xi \\ 0 & g \end{pmatrix} : \mathcal{C}(f)_r = D_r \oplus C_{r-1} \to \mathcal{C}(f')_r = D'_r \oplus C'_{r-1} \ ,$$

then $g : C \to C'$ and $g' : D \to D'$ are chain maps, and a chain homotopy $h : g'f \simeq f'g$ can be defined by the formula

$$h = (-)^{r+1}\xi : C_r \to D'_{r+1} \ .$$

If both g and g' are chain equivalences, then the next Proposition assures us that G itself is a chain equivalence.

Proposition. *If both g and g' are chain equivalences in the triad above, then $(g, g'; h) : \mathcal{C}(f) \to \mathcal{C}(f')$ is also a chain equivalence.*

If all the chain complexes are free, then $\mathcal{C}((g, g'; h))$ is chain contractible if and only if it is acyclic, and the above can be deduced using homology exact sequences and 5-lemma. We give a proof which does not use homology.

Proof: Consider the following triad:

$$\begin{tikzcd} C \arrow[r, "-g"] \arrow[d, "-f"'] \arrow[dr, squiggly, "h"] & C' \arrow[d, "f'"] \\ D \arrow[r, "g'"'] & D' \end{tikzcd}$$

A direct calculation shows that $\mathcal{C}((g, g'; h)) \cong \mathcal{C}((-f, f'; h))$. Since $\mathcal{C}(-g)$ and $\mathcal{C}(g')$ are both chain contractible, $(-f, f'; h) : \mathcal{C}(-g) \to \mathcal{C}(g')$ is a chain equivalence. Therefore $(g, g'; h)$ is also a chain equivalence. $\square$

From the triad Γ above, we can produce another triad

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, squiggly, "-h"] & D \arrow[d, "f'"] \\ C' \arrow[r, "g'"'] & D' \end{tikzcd}$$

and this induces a chain map $(f, f'; -h) : \mathcal{C}(g) \to \mathcal{C}(g')$. The algebraic mapping cone $\mathcal{C}(-(f, f'; -h))$ is denoted $\mathcal{C}(\Gamma)$ [Ranicki]. It is isomorphic to $\mathcal{C}((g, g'; h))$ and $\mathcal{C}((-f, f'; h))$

Let $f : C \to D$ be a chain map of chain complexes in $\mathbb{A}$, then D is a subcomplex of the algebraic mapping cone $\mathcal{C}(f)$. Let $i : D \to \mathcal{C}(f)$ denote the inclusion map. Note that the quotient $\mathcal{C}(f)/D$ is equal to the suspension SC of C. We can extend the sequence $C \xrightarrow{f} D \xrightarrow{i} \mathcal{C}(f)$ to the right by taking the algebraic mapping cone of i:

$$C \xrightarrow{f} D \xrightarrow{i} \mathcal{C}(f) \xrightarrow{j} \mathcal{C}(i) \ ,$$

where j is the inclusion map.

Proposition. *$\mathcal{C}(i)$ is chain equivalent to $\mathcal{C}(f)/D = SC$.*

Proof: Note that $\mathcal{C}(i)$ and SC can be viewed as the algebraic mapping cones of

$$\begin{aligned} F = \begin{pmatrix} f \\ 0 \end{pmatrix} &: C \longrightarrow \mathcal{C}(1_D : D \to D)\ , \\ 0 &: C \longrightarrow 0\ , \end{aligned}$$

respectively. These fit into the following triad of chain complexes:

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "1_C"'] \arrow[dr, squiggly, "0"] & \mathcal{C}(1_D) \arrow[d, "0"] \\ C \arrow[r, "0"'] & 0 \end{tikzcd}$$

whose vertical arrows are both chain equivalences. Therefore the chain map $(1_C, 0; 0)$ gives the desired chain equivalence. $\square$

Since it will be necessary to have the explicit formula for the chain equivalences, we write them down here. The following matrices give a chain equivalence and a homotopy inverse:

$$\begin{aligned} \alpha = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} &: \mathcal{C}(i)_r = D_r \oplus C_{r-1} \oplus D_{r-1} \longrightarrow C_{r-1} = (SC)_r, \\ \beta = \begin{pmatrix} 0 \\ 1 \\ -f \end{pmatrix} &: (SC)_r = C_{r-1} \longrightarrow D_r \oplus C_{r-1} \oplus D_{r-1} = \mathcal{C}(i)_r. \end{aligned}$$

The chain map β is induced by the triad :

$$\begin{tikzcd} C \arrow[r, "0"] \arrow[d, "1_C"'] \arrow[dr, squiggly, "h"] & 0 \arrow[d, "0"] \\ C \arrow[r, "F"'] & \mathcal{C}(1_D) \end{tikzcd}$$

where the chain homotopy $h : 0 \simeq \begin{pmatrix} f \\ 0 \end{pmatrix} : C \to \mathcal{C}(1_D)$ is defined by

$$\begin{pmatrix} 0 \\ (-)^r f \end{pmatrix} \ : \ C_r \longrightarrow D_{r+1} \oplus D_r = \mathcal{C}(1_D)_{r+1}$$

The composite $\alpha\beta$ is obviously 1_{SC}. The other composite $\beta\alpha$ is given by the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -f & 0 \end{pmatrix} \; : \; D_r \oplus C_{r-1} \oplus D_{r-1} \longrightarrow D_r \oplus C_{r-1} \oplus D_{r-1},$$

and it is chain homotopic to $1_{\mathcal{C}(i)}$; the following matrix defines a desired chain homotopy:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (-)^r & 0 & 0 \end{pmatrix} \; : \; D_r \oplus C_{r-1} \oplus D_{r-1} \longrightarrow D_{r+1} \oplus C_r \oplus D_r.$$

We can continue this process to get a sequence

$$C \xrightarrow{f} D \xrightarrow{i} \mathcal{C}(f) \xrightarrow{j} \mathcal{C}(i) \xrightarrow{k} \mathcal{C}(j) \xrightarrow{l} \mathcal{C}(k) \xrightarrow{m} \mathcal{C}(l) \longrightarrow \ldots,$$

where k, l, ... are inclusion maps. Again , $\mathcal{C}(j)$, $\mathcal{C}(k)$, $\mathcal{C}(l)$, ... are chain equivalent to $\mathcal{C}(i)/\mathcal{C}(f) = SD$, $\mathcal{C}(j)/\mathcal{C}(i) = S\mathcal{C}(f)$, $\mathcal{C}(k)/\mathcal{C}(j) = S\mathcal{C}(i) \simeq S^2C$, ..., and we obtain a sequence

$$C \xrightarrow{f} D \xrightarrow{i} \mathcal{C}(f) \xrightarrow{p} SC \xrightarrow{Sf} SD \xrightarrow{Si} S\mathcal{C}(f) \xrightarrow{Sp} S^2C \xrightarrow{S^2f} \ldots.$$

In the case of the inclusion map $i : C' \to C$ from a subcomplex C' of C into C, C is the algebraic mapping cone $\mathcal{C}(\rho)$ of some chain map $\rho : \Omega(C/C') \to C'$, and we obtain a sequence

$$\Omega(C/C') \xrightarrow{\rho} C' \xrightarrow{i} C \xrightarrow{p} C/C' \xrightarrow{S\rho} SC' \xrightarrow{Si} SC \xrightarrow{Sp} S(C/C') \xrightarrow{S^2\rho} \ldots.$$

The chain equivalence

$$\beta = \begin{pmatrix} 0 \\ 1 \\ -\rho \end{pmatrix} : (C/C')_r \to C'_r \oplus (C/C')_r \oplus (C/C')_{r-1} = \mathcal{C}(i)_r$$

from C/C' to $\mathcal{C}(i)$ will be used repeatedly later.

2. Quadratic structures on complexes and chain maps

In this section we review the basics of Ranicki's theory of quadratic complexes in an additive category with involution [Additive L-theory].

Definition. An *involution* on an additive category $\mathbb{A}$ is a contravariant functor

$$* \ : \ \mathbb{A} \to \mathbb{A} \ ; \ \begin{cases} M \ \mapsto M^* \\ (f : M \to N) \mapsto (f^* : N^* \to M^*) \end{cases}$$

together with a natural equivalence

$$e \ : \ 1_{\mathbb{A}} \to ** \ : \ \mathbb{A} \to \mathbb{A} \ ; \ M \ \mapsto \ (e(M) : M \to M^{**})$$

such that for every object M in $\mathbb{A}$

$$e(M)^* e(M^*) = 1_{M^*} \ : \ M^* \to M^{***} \to M^* \ .$$

We shall use $e(M)$ to identify $M^{**} = M$.

Examples. Let R denote a ring with involution, *i.e.* an associative ring with 1 together with a function

$$^- : \ R \to R \ ; \ a \mapsto \bar{a}$$

such that

$$\overline{a+b} = \bar{a} + \bar{b} \ , \ \ \overline{ab} = \bar{b}\bar{a} \ , \ \ \bar{\bar{a}} = a \ , \ \ \bar{1} = 1 \ \in R \ \ (a, b \in R) \ .$$

Let us consider a subcategory $\mathbb{A}$ of the category of the left R-modules and the R-module morphisms. We list requirements for $\mathbb{A}$ below, while giving several definitions. (The part printed *in italics* are not part of the requirement. They state the general fact or definiton.)

1. 0 and R are objects in $\mathbb{A}$.
2. If M and N are objects in $\mathbb{A}$, then so is $M \oplus N$, and the inclusion maps $M \to M \oplus N$, $N \to M \oplus N$ and the projection maps $M \oplus N \to M$, $M \oplus N \to N$ are morphisms in $\mathbb{A}$.
3. For each pair of objects M, N of $\mathbb{A}$, the set of the morphisms $\mathrm{Hom}(M, N)$ from M to N is an abelian group. In particular, if $f : M \to N$ is a morphism, then so is $-f : M \to N$.
4. For each $a \in R$ and each $f \in \mathrm{Hom}(M, R)$, the left R-module morphism af : $M \to R$ which sends $x \in M$ to $f(x) \cdot \bar{a}$ is a morphism of $\mathbb{A}$, and with respect to this left R-module structure $\mathrm{Hom}(M, R)$ is an object of $\mathbb{A}$. This is called the *dual* of M and is denoted M^*.

5. For each morphism $f : M \to N$ and $g \in N^*$, gf is an element of M^*. Then $f : M \to N$ *induces a left R-module morphism*

$$f^* \; : \; N^* \to M^* \; ; \; g \mapsto gf \; .$$

We require this to be an element of $\mathrm{Hom}(N^*, M^*)$, and it is called the *dual* of f.

6. For each object M in $\mathbb{A}$ and each element $x \in M$, *there is a left R-module morphism*

$$M^* \to R \; ; \; h \mapsto \overline{h(x)} \; .$$

We require that this be an element of M^{**} and that the correspondence

$$M \to M^{**} \; ; \; x \mapsto (h \mapsto \overline{h(x)}) \quad (h \in M^*)$$

be an isomorphism in the category $\mathbb{A}$.

Then $\mathbb{A}$ is an additive category with involution in the sense above. Each of the following satisfy these requiremets:
(a) the category of finitely generated free R-modules,
(b) the category of finitely generated projective R-modules,
(c) the category of based free R-modules and locally finite R-module morphisms,
where an R-module morphism $f : \bigoplus Rx_\alpha \to \bigoplus Ry_\beta$ between based free R-modules is *locally finite* if for every β there are only finitely many α's such that y_β appears with non-trivial coefficient in $f(x_\alpha)$.

In the rest of the section, we fix an additive category $\mathbb{A}$ with involution.

For a chain complex C in $\mathbb{A}$, let C^* be the chain complex defined by

$$d_{C^*} = (d_C)^* : (C^*)_r = C^{-r} = (C_{-r})^* \to (C^*)_{r-1} = C^{-r+1},$$

and let C^{n-*} ($n \in \mathbb{Z}$) be the chain complex defined by

$$d_{C^{n-*}} = (-)^r (d_C)^* : (C^{n-*})_r = C^{n-r} \to (C^{n-*})_{r-1} = C^{n-r+1}.$$

Proposition. (1) *If $f : C \to D$ is a chain map of chain complexes in $\mathbb{A}$, then so is $f^* : D^{n-*} \to C^{n-*}$. If f is a chain equivalence, then so is f^*.*
(2) *C^{n+1-*} is isomorphic to $S(C^{n-*})$.*

Proof: (1) Immediate from the definiton.
(2) An isomorphism is defined by

$$C^{n+1-r} \ni x \mapsto (-)^{n-r} x \in C^{n-(r-1)} \; .$$

$\square$

Now we fix $\epsilon = \pm 1$. Given a finite-dimensional chain complex C in $\mathbb{A}$ let the generator $T \in \mathbb{Z}_2$ act on $\mathrm{Hom}(C^*, C)$ by the involution

$$T_\epsilon : \mathrm{Hom}(C^p, C_q) \to \mathrm{Hom}(C^q, C_p) \quad ; \qquad f \mapsto (-)^{pq}\epsilon f^* \ .$$

Define a $\mathbb{Z}$-module chain complex $W_\% C$ by

$$W_\% C = W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathrm{Hom}(C^*, C),$$

where W is the standard free $\mathbb{Z}[\mathbb{Z}_2]$-resolution of $\mathbb{Z}$

$$W : \ldots \to \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \to 0.$$

$(W_\% C)_n$ is isomorphic to $\sum_{s \geq 0} \mathrm{Hom}(C^*, C)_{n-s}$ and an n-chain of $W_\% C$ can be viewed as a collection

$$\psi = (\psi_s \in \mathrm{Hom}(C^{n-r-s}, C_r))_{s \geq 0}.$$

Its boundary is

$$\big(d\psi_s + (-)^r \psi_s d^* + (-)^{n-s-1}(\psi_{s+1} + (-)^{s+1} T\psi_{s+1}) : C^{n-r-s-1} \to C_r \quad (r \in \mathbb{Z})\big)_{s \geq 0} .$$

$W_\% C$ is a $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex by the action: $T\psi = (T\psi_s)$. In particular, if ψ is a cycle then so is $T\psi$. The set of n-cycles of $W_\% C$ is denoted $\mathcal{Z}_n C$. It is a $\mathbb{Z}[\mathbb{Z}_2]$-module.

Definition. An element ψ of $\mathcal{Z}_n C$ is called an *n-dimensional quadratic structure on C*. It induces a chain map in $\mathbb{A}$

$$\mathcal{D}_\psi = (1+T)\psi_0 \ : \ C^{n-*} \to C \ ,$$

which is called the *duality map*. The quadratic structure ψ is said to be *Poincaré* if $\mathcal{D}_\psi$ is a chain equivalence. A pair (C, ψ) of an n-dimensional complex C in $\mathbb{A}$ and an n-dimensional quadratic (Poincaré) structure ψ is called an *n-dimensional quadratic (Poincaré) complex in* $\mathbb{A}$.

Remark. Homologous n-dimensional quadratic structures ψ, $\psi' \in \mathcal{Z}_n C$ on C induce chain homotopic duality maps $\mathcal{D}_\psi$, $\mathcal{D}_{\psi'}$. If $\chi \in (W_\% C)_{n+1}$ is a chain such that $d\chi = \psi' - \psi$, then $(1+T)\chi_0 : C^{n-r} \to C_{r+1}$ defines a desired chain homotopy between $\mathcal{D}_\psi$ and $\mathcal{D}_{\psi'}$.

A chain map $f : C \longrightarrow D$ in $\mathbb{A}$ induces a $\mathbb{Z}[\mathbb{Z}_2]$-module chain map

$$\mathrm{Hom}(f^*, f) : \mathrm{Hom}(C^*, C) \to \mathrm{Hom}(D^*, D) \quad ; \quad \phi \mapsto f\phi f^*,$$

and hence also a $\mathbb{Z}[\mathbb{Z}_2]$-module chain map $f_\% : W_\% C \to W_\% D$;

$$\psi = (\psi_s) \mapsto f_\% \psi = (f \psi_s f^*).$$

Define a $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex $W_\% f$ by

$$W_\% f = \mathcal{C}(f_\% : W_\% C \to W_\% D).$$

The boundary of an $(n+1)$-chain $(\delta\psi, \psi) \in (W_\% D)_{n+1} \oplus (W_\% C)_n$ of $W_\% f$ is

$$\begin{aligned}\Big(&(d(\delta\psi_s) + (-)^r(\delta\psi_s)d^* + (-)^{n-s}(\delta\psi_{s+1} + (-)^{s+1}T\delta\psi_{s+1}) + (-)^n f\psi_s f^* \\ &\qquad : D^{n-r-s} \to D_r \quad (r \in \mathbb{Z}))_{s\geq 0}, \\ &(d\psi_s + (-)^r \psi_s d^* + (-)^{n-s-1}(\psi_{s+1} + (-)^{s+1}T\psi_{s+1}) \\ &\qquad : C^{n-r-s-1} \to C_r \quad (r \in \mathbb{Z}))_{s\geq 0}\Big).\end{aligned}$$

The set of n-cycles of $W_\% f$ is denoted $\mathcal{Z}_n f$. It is a $\mathbb{Z}[\mathbb{Z}_2]$-module.

Definition. An element $(\delta\psi, \psi)$ of $\mathcal{Z}_{n+1} f$ is called an $(n+1)$-*dimensional quadratic structure on* f. The *duality map* $\mathcal{D}_{(\delta\psi,\psi)} : D^{n+1-*} \to \mathcal{C}(f)$ is a chain map in $\mathbb{A}$ defined by the matrix:

$$\mathcal{D}_{(\delta\psi,\psi)} = \begin{pmatrix} (1+T)\delta\psi_0 \\ (-)^{n+1-r}(1+T)\psi_0 f^* \end{pmatrix} \ : \ D^{n+1-r} \to \mathcal{C}(f)_r = D_r \oplus C_{r-1}$$

The quadratic structure $(\delta\psi, \psi)$ is said to be *Poincaré* if $\mathcal{D}_{(\delta\psi,\psi)}$ is a chain equivalence. A pair $(f : C \to D, (\delta\psi, \psi))$ of a chain map f in $\mathbb{A}$ from an n-dimensional chain complex C to an $(n+1)$-dimensional chain complex D and an $(n+1)$-dimensional quadratic (Poincaré) structure $(\delta\psi, \psi)$ is called an $(n+1)$-*dimensional quadratic (Poincaré) pair in* $\mathbb{A}$.

Remarks. (1) An $(n+1)$-dimensional quadratic structure $(\delta\psi, \psi)$ on $f : C \to D$ is Poincaré if and only if the T-dual $\bar{\mathcal{D}}_{(\delta\psi,\psi)} : \mathcal{C}(f)^{n+1-*} \to D$;

$$\bar{\mathcal{D}}_{(\delta\psi,\psi)} = ((1+T)\delta\psi_0, f(1+T)\psi_0) \ : \ D^{n+1-r} \oplus C^{n-r} \to D \ .$$

of the duality map $\mathcal{D}_{(\delta\psi,\psi)} : D^{n+1-*} \to \mathcal{C}(f)$ is a chain equivalence.
(2) If $(f : C \to D, (\delta\psi, \psi))$ is an $(n+1)$-dimensional quadratic (Poincaré) pair in $\mathbb{A}$, then its *boundary* (C, ψ) is an n-dimensional quadratic (Poincaré) complex in $\mathbb{A}$.
(3) Homologous elements $(\delta\psi, \psi)$, $(\delta\psi', \psi') \in \mathcal{Z}_{n+1} f$ induce chain homotopic duality maps $\mathcal{D}_{(\delta\psi,\psi)}$, $\mathcal{D}_{(\delta\psi',\psi')}$. If $(\delta\chi, \chi) \in (W_\% f)_{n+2}$ is a chain whose boundary is $(\delta\psi', \psi') - (\delta\psi, \psi)$, then

$$\begin{pmatrix} (1+T)\delta\chi_0 \\ (-)^{n+1-r}(1+T)\chi_0 f^* \end{pmatrix} \ : \ D^{n+1-r} \to \mathcal{C}(f)_r = D_{r+1} \oplus C_{r-1}$$

defines a desired chain homotopy.

From given quadratic structures, we can construct other structures in several ways. Firstly, the T-*duals* of quadratic structures are also quadratic structures: If $\psi = \{\psi_s\}$ is an n-dimensional quadratic structure on C, then so is $T\psi = \{T\psi_s\}$; if $(\delta\psi, \psi)$ is an $(n+1)$-dimensional quadratic structure on $f : C \to D$, then so is $T(\delta\psi, \psi) = (T\delta\psi, T\psi) = \{T\delta\psi_s, T\psi_s\}$. In general, the T-dual is not homologous to the original structure. Secondly, we can consider $\hat{T}$-*duals* of quadratic structures. For $\theta = \{\theta_s\}$, define $\hat{T}\theta = \{\hat{T}\theta_s\}$ by

$$\hat{T}\theta_0 = T\theta_0 \quad \text{and} \qquad \hat{T}\theta_s = -\theta_s \quad \text{for } s > 0\ .$$

If $\psi = \{\psi_s\}$ is an n-dimensional quadratic structure on C, then so is $\hat{T}\psi = \{\hat{T}\psi_s\}$; if $(\delta\psi, \psi)$ is an $(n+1)$-dimensional quadratic structure on $f : C \to D$, then so is $\hat{T}(\delta\psi, \psi) = (\hat{T}\delta\psi, \hat{T}\psi) = \{\hat{T}\delta\psi_s, \hat{T}\psi_s\}$.

Proposition. *The $\hat{T}$-dual is homologous to the original quadratic structure.*

Proof. Suppose $(\delta\psi, \psi)$ is an $(n+1)$-dimensional quadratic structure on f. Define $\delta\chi_s : D^{n+2-r-s} \to D_r$ and $\chi_s : C^{n+1-r-s} \to C_r$ by:

$$\begin{aligned} \delta\chi_0 &= 0\ , & \delta\chi_s &= (-)^{n-s}\delta\psi_{s-1} \quad \text{for } s > 0, \\ \chi_0 &= 0\ , & \chi_s &= (-)^{n-1-s}\psi_{s-1} \quad \text{for } s > 0\ . \end{aligned}$$

Then a direct calculation shows that $d(\delta\chi, \chi) = (\delta\psi, \psi) - \hat{T}(\delta\psi, \psi)$. $\square$

We describe the union operation of adjoing pairs. Let $f : C \to D'$, $g : C \to D''$ be chain maps in $\mathbb{A}$ and suppose we are given two $(n+1)$-cycles $(\delta\psi', -\psi) \in \mathcal{Z}_{n+1}f$, $(\delta\psi'', \psi) \in \mathcal{Z}_{n+1}g$. Let $D = D' \cup_C D''$ denote the algebraic mapping cone $\mathcal{C}(\begin{pmatrix} f \\ g \end{pmatrix})$ and define the *union* $\delta\psi' \cup_\psi \delta\psi'' \in \mathcal{Z}_{n+1}D$ *along* ψ by

$$(\delta\psi' \cup_\psi \delta\psi'')_s = \begin{pmatrix} \delta\psi'_s & 0 & 0 \\ (-)^{n-r}\psi_s f^* & (-)^{n+1-r-s}T\psi_{s+1} & 0 \\ 0 & (-)^s g\psi_s & \delta\psi''_s \end{pmatrix}$$
$$: D^{n+1-r-s} = D'^{n+1-r-s} \oplus C^{n-r-s} \oplus D''^{n+1-r-s} \to D_r = D'_r \oplus C_{r-1} \oplus D''_r\ .$$

whose T-dual is expressed by the matrix:

$$T(\delta\psi' \cup_\psi \delta\psi'')_s = \begin{pmatrix} T\delta\psi'_s & (-)^{s+1}fT\psi_s & 0 \\ 0 & (-)^{n-r-s}\psi_{s+1} & (-)^{n+1-r}T\psi_s g^* \\ 0 & 0 & T\delta\psi''_s \end{pmatrix}$$
$$: D^{n+1-r-s} = D'^{n+1-r-s} \oplus C^{n-r-s} \oplus D''^{n+1-r-s} \to D_r = D'_r \oplus C_{r-1} \oplus D''_r\ ,$$

If $(\delta\psi', -\psi)$ and $(\delta\psi'', \psi)$ are Poincaré then the union $\delta\psi' \cup_\psi \delta\psi''$ is also Poincaré. This will be proved in a more general context later and we only record the duality map for the union:

$$(1+T)(\delta\psi' \cup_\psi \delta\psi'')_0 = \begin{pmatrix} (1+T)\delta\psi'_0 & -fT\psi_0 & 0 \\ (-)^{n-r}\psi_0 f^* & (-)^{n-r}(1-T)\psi_1 & (-)^{n+1-r}T\psi_0 g^* \\ 0 & g\psi_0 & (1+T)\delta\psi''_0 \end{pmatrix}$$
$$: D^{n+1-r} = D'^{n+1-r} \oplus C^{n-r} \oplus D''^{n+1-r} \to D_r = D'_r \oplus C_{r-1} \oplus D''_r ,$$

The pair $(D, \delta\psi' \cup_\psi \delta\psi'')$ is called the *union* of the pairs $(f : C \to D', (\delta\psi', -\psi))$ and $(g : C \to D'', (\delta\psi'', \psi))$ along (C, ψ).

If we switch the order and take the union of $(\delta\psi'', \psi)$ and $(\delta\psi', -\psi)$ along $-\psi$, then we get another element represented by a matrix

$$(\delta\psi'' \cup_{-\psi} \delta\psi')_s = \begin{pmatrix} \delta\psi'_s & (-)^{s+1} f\psi_s & 0 \\ 0 & (-)^{n-r-s}T\psi_{s+1} & (-)^{n+1-r}\psi_s g^* \\ 0 & 0 & \delta\psi''_s \end{pmatrix}$$
$$: D^{n+1-r-s} = D'^{n+1-r-s} \oplus C^{n-r-s} \oplus D''^{n+1-r-s} \to D_r = D'_r \oplus C_{r-1} \oplus D''_r ,$$

after reordering the direct summands. This element is homologous to $\delta\psi' \cup_\psi \delta\psi''$, because the difference

$$\delta\psi'' \cup_{-\psi} \delta\psi' - \delta\psi' \cup_\psi \delta\psi''$$

is the boundary of an $(n+2)$-chain $\theta \in W_\% D$ defined by:

$$\theta_s = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (-)^{r+s}\psi_s & 0 \\ 0 & 0 & 0 \end{pmatrix} : D^{n+2-r} = D'^{n+2-s-r} \oplus C^{n+1-s-r} \oplus D''^{n+2-s-r}$$
$$\to D_r = D'_r \oplus C_{r-1} \oplus D''_r .$$

Also note the identity $\delta\psi'' \cup_{-\psi} \delta\psi' = T(T\delta\psi' \cup_{T\psi} T\delta\psi'')$.

Definition. A *cobordism* between n-dimensional quadratic complexes (C, ψ), (C', ψ') in $\mathbb{A}$ is an $(n+1)$-dimensional quadratic pair

$$(\, f \quad f' \,) : C \oplus C' \to D, (\delta\psi, \psi \oplus -\psi') \in W_\%((\, f \quad f' \,))$$

with boundary $(C \oplus C', \psi \oplus -\psi')$. Such a cobordism is *Poincaré* if the pair above is Poincaré.

Remarks. (1) Suppose (C, ψ) is an n-dimensional quadratic (Poincaré) complex and ψ' is an element of $\mathcal{Z}_n C$ homologous to ψ, *i.e.* $d_{W_\% C}\delta\psi = \psi' - \psi$ for some $\delta\psi$, then

$$((\,1 \quad 1\,) : C \oplus C \to C, \quad ((-)^n \delta\psi, \psi \oplus -\psi'))$$

is a (Poincaré) cobordism between (C, ψ) and (C, ψ').
(2) The union operarion described above generalizes to the case of adjoining cobordisms. Given two consecutive cobordisms

$$\begin{aligned}&(f', f) : C' \oplus C \to D', (\delta\psi', \psi' \oplus -\psi)\\ &(g, g'') : C \oplus C'' \to D'', (\delta\psi'', \psi \oplus -\psi'')\ ,\end{aligned}$$

define D to be the algebraic mapping cone $\mathcal{C}(\begin{pmatrix} f \\ g \end{pmatrix})$ and define $\delta\psi' \cup_\psi \delta\psi''$ by the matrix

$$(\delta\psi' \cup_\psi \delta\psi'')_s = \begin{pmatrix} \delta\psi'_s & 0 & 0 \\ (-)^{n-r}\psi_s f^* & (-)^{n+1-r-s}T\psi_{s+1} & 0 \\ 0 & (-)^s g\psi_s & \delta\psi''_s \end{pmatrix}$$
$$: D^{n+1-r-s} = D'^{n+1-r-s} \oplus C^{n-r-s} \oplus D''^{n+1-r-s} \to D_r = D'_r \oplus C_{r-1} \oplus D''_r,$$

as in the case of the union of pairs. Let $i_{D'} : D' \to D$ and $i_{D''} : D'' \to D$ be the inclusion maps, then

$$(\,i_{D'}f' \quad i_{D''}g''\,) : C' \oplus C'' \to D,\ (\delta\psi' \cup_\psi \delta\psi'', \psi' \oplus -\psi'')$$

is a cobordism between (C', ψ') and (C'', ψ''). This is the *union* of the two cobordisms along (C, ψ). If the two cobordisms above are Poincaré, then their union is also Poincaré (see the next section).

Proposition. *Poincaré cobordism is an equivalence relation on the set of n-dimensional quadratic Poincaré complexes in $\mathbb{A}$. The Poincaré cobordism classes define an abelian group, the n-dimensional quadratic L-group of $\mathbb{A}$, $L_n(\mathbb{A})$ $(n \geq 0)$, with addition and inverses by*

$$[C, \psi] + [C', \psi'] = [C \oplus C', \psi \oplus \psi'], \qquad -[C, \psi] = [C, -\psi].$$

Proof: Use the construction given in the remarks above. $\square$

3. Quadratic structures on triads

In this section we discuss quadratic structures on triads of chain complexes, and also discuss the union operation of such structures. Before giving the definition, we need to study several properties of the $\mathbb{Z}[\mathbb{Z}_2]$-module chain map $f_\% : W_\% C \to W_\% D$ induced from a chain map $f : C \to D$ in $\mathbb{A}$. Here $\mathbb{A}$ is a fixed additive category with involution and $\epsilon = \pm 1$ is chosen.

Proposition. *A chain homotopy $k : f \simeq g : C \to D$ in $\mathbb{A}$ induces a $\mathbb{Z}$-module chain homotopy $k_\% : f_\% \simeq g_\% : W_\% C \to W_\% D$;*

$$k_\% : (W_\% C)_n \longrightarrow (W_\% D)_{n+1};$$
$$(\psi_s)_{s\geq 0} \mapsto (k\psi_s f^* + (-)^r g\psi_s k^* + (-)^{n-r} k(T\psi_{s+1})k^* : D^{n-s+1-r} \to D_r)_{s\geq 0}.$$

Remark. Note that $k_\%$ depends not only on k but also on f and g and that it is only a $\mathbb{Z}$-module chain homotopy.

Proof. This is proved by a straightforward calculation. In fact,

$$\begin{aligned}
&(d_{W_\% D} k_\%)(\psi)_s = d(k\psi_s f^* + (-)^{r+1} g\psi_s k^* + (-)^{n-r-1} kT\psi_{s+1}k^*) \\
&\quad + (-)^r (k\psi_s f^* + (-)^r g\psi_s k^* + (-)^{n-r} kT\psi_{s+1}k^*)d^* \\
&\quad + (-)^{n-s}\{(k\psi_{s+1}f^* + (-)^r g\psi_{s+1}k^* + (-)^{n-r} kT\psi_{s+2}k^*) \\
&\quad + (-)^{s+1}((-)^r fT\psi_{s+1}k^* + kT\psi_{s+1}g^* + (-)^{n-r-1} k\psi_{s+2}k^*)\} \ , \\
&(k_\% d_{W_\% C})(\psi)_s = k(d\psi_s + (-)^{r-1}\psi_s d^* + (-)^{n-s-1}(\psi_{s+1} + (-)^{s+1}T\psi_{s+1}))f^* \\
&\quad + (-)^r g(d\psi_s + (-)^r \psi_s d^* + (-)^{n-s-1}(\psi_{s+1} + (-)^{s+1}T\psi_{s+1}))k^* \\
&\quad + (-)^{n-1-r} k(dT\psi_{s+1} + (-)^{r-1}T\psi_{s+1}d^* + (-)^{n-s}(T\psi_{s+2} + (-)^{s+2}\psi_{s+2}))k^* \\
&\qquad : \ D^{n-s-r} \longrightarrow D_r \ ,
\end{aligned}$$

and we obtain $(d_{W_\% D} k_\% + k_\% d_{W_\% C})(\psi)_s = g\psi_s g^* - f\psi_s f^*$. □

Proposition. If $f : C \to D$, $g : D \to E$ are chain maps in $\mathbb{A}$, then $(gf)_\% = g_\% f_\% \ : \ W_\% C \to W_\% E$.

Proof. Obvious. □

Now consider a triad Γ in $\mathbb{A}$:

$$\begin{tikzcd}
C \arrow[r, "f'"] \arrow[d, "f''"'] \arrow[dr, squiggly, "k"] & D' \arrow[d, "g'"] \\
D'' \arrow[r, "g''"'] & E
\end{tikzcd} \ .$$

It induces a triad of $\mathbb{Z}[\mathbb{Z}_2]$-module chain complexes

$$\begin{tikzcd} W_\% C \arrow[r, "f'_\%"] \arrow[d, "f''_\%"'] \arrow[dr, squiggly, "k_\%"] & W_\% D' \arrow[d, "g'_\%"] \\ W_\% D'' \arrow[r, "g''_\%"'] & W_\% E \end{tikzcd} .$$

in which $f'_\%$,$f''_\%$,$g'_\%$,$g''_\%$ are $\mathbb{Z}[\mathbb{Z}_2]$-module chain maps but $k_\% : g'_\% f'_\% \simeq g''_\% f''_\%$ is only a $\mathbb{Z}$-module chain homotopy. $W_\%\Gamma$ will denote the $\mathbb{Z}$-module chain complex $\mathcal{C}((f''_\%, g'_\%; k_\%) : W_\% f' \to W_\% g'')$. A typical $(n+2)$-chain is a quadruple

$$(\delta\chi, \delta\psi', \delta\psi'', \psi) \in (W_\% E)_{n+2} \oplus (W_\% D')_{n+1} \oplus (W_\% D'')_{n+1} \oplus (W_\% C)_n \ ,$$

and its boundary is:

$$\begin{aligned}
&(d\delta\chi_s + (-)^r \delta\chi_s d^* + (-)^{n-s+1}(\delta\chi_{s+1} + (-)^{s+1} T\delta\chi_{s+1}) + (-)^{n+1} g'\delta\psi'_s g'^* \\
&\qquad + (-)^{n+1} g''\delta\psi''_s g''^* + k\psi_s f'^* g'^* + (-)^r g'' f''\psi_s k^* + (-)^{n-r} k(T\psi_{s+1})k^* \\
&\qquad\qquad : E^{n+1-s-r} \to E_r \qquad (r \in \mathbb{Z}), \\
&d\delta\psi'_s + (-)^r \delta\psi'_s d^* + (-)^{n-s}(\delta\psi'_{s+1} + (-)^{s+1} T\delta\psi'_{s+1}) + (-)^n f'\psi_s f'^* \\
&\qquad\qquad : D'^{n-s-r} \to D'_r \qquad (r \in \mathbb{Z}), \\
&d\delta\psi''_s + (-)^r \delta\psi''_s d^* + (-)^{n-s}(\delta\psi''_{s+1} + (-)^{s+1} T\delta\psi''_{s+1}) - (-)^n f''\psi_s f''^* \\
&\qquad\qquad : D''^{n-s-r} \to D''_r \qquad (r \in \mathbb{Z}), \\
&d\psi_s + (-)^r \psi_s d^* + (-)^{n-s-1}(\psi_{s+1} + (-)^{s+1} T\psi_{s+1}) \\
&\qquad\qquad : C^{n-s-r-1} \to C_r \qquad (r \in \mathbb{Z}))_{s \geq 0}.
\end{aligned}$$

$\mathcal{Z}_n\Gamma$ denotes the set of n-cycles of $W_\%\Gamma$.

Note that $(\delta\psi', \psi) \in \mathcal{Z}_{n+1} f'$ and $(\delta\psi'', -\psi) \in \mathcal{Z}_{n+1} f''$ and that we can glue these. Let $(D' \cup_C D'', \chi = \delta\psi' \cup_{-\psi} \delta\psi'')$ be their union along $(C, -\psi)$, and define a chain map $g : D' \cup_C D'' \to E$ in $\mathbb{A}$ by

$$g = \begin{pmatrix} g' & (-)^r k & -g'' \end{pmatrix} : D'_r \oplus C_{r-1} \oplus D''_r \to E_r.$$

Then $(\delta\chi, \chi)$ is an $(n+2)$-dimensional quadratic structure on $g : D' \cup_C D'' \to E$. The algebraic mapping cone $\mathcal{C}(g)$ is equal to the chain complex $\mathcal{C}(\Gamma)$ introduced in the first section; its boundary map is expressed by the following matrix:

$$d_{\mathcal{C}(\Gamma)} = \begin{pmatrix} d_E & (-)^{r-1} g' & (-)^r g'' & k \\ 0 & d_{D'} & 0 & (-)^r f' \\ 0 & 0 & d_{D''} & (-)^r f'' \\ 0 & 0 & 0 & d_C \end{pmatrix}$$
$$: \mathcal{C}(\Gamma)_r = E_r \oplus D'_{r-1} \oplus D''_{r-1} \oplus C_{r-2} \to \mathcal{C}(\Gamma)_{r-1} = E_{r-1} \oplus D'_{r-2} \oplus D''_{r-2} \oplus C_{r-3}.$$

Definitions. (1) An element $\Psi = (\delta\chi, \delta\psi', \delta\psi'', \psi) \in \mathcal{Z}_{n+2}\Gamma$ is called an $(n+2)$-*dimensional quadratic structure on the triad* Γ. The *duality map* $\mathcal{D}_\Psi : E^{n+2-*} \to \mathcal{C}(\Gamma)$ is the chain map in $\mathbb{A}$ defined by the matirix:

$$\mathcal{D}_\Psi = \begin{pmatrix} (1+T)\delta\chi_0 + (-)^{r-1}k\psi_0 k^* \\ (-)^{n+2-r}(1+T)\delta\psi'_0 g'^* + f'(1+T)\psi_0 k^* \\ (-)^{n+1-r}(1+T)\delta\psi''_0 g''^* \\ (1+T)\psi_0 f''^* g''^* \end{pmatrix}$$
$$: E^{n+2-r} \to \mathcal{C}(\Gamma)_r = E_r \oplus D'_{r-1} \oplus D''_{r-1} \oplus C_{r-2}\ .$$

(2) A quadratic structure Ψ is *Poincaré* if the $(n+1)$-dimensional quadratic structures $(\delta\psi', \psi)$ on f' and $(\delta\psi'', -\psi)$ on f'' are both Poincaré, and the duality map $\mathcal{D}_\Psi$ is a chain equivalence.
(3) A pair (Γ, Ψ) of a triad Γ as above in $\mathbb{A}$ and an $(n+2)$-dimensional quadratic (Poincaré) structure Ψ on Γ is said to be an $(n+2)$-*dimensional quadratic (Poincaré) triad in* $\mathbb{A}$ if C is n-dimensional, D' and D'' are $(n+1)$-dimensional, E is $(n+2)$-dimensional.
(4) The pair $(g : D' \cup_C D'' \to E, (\delta\chi, \chi))$ introduced above is called the *quadratic pair associated with* (Γ, Ψ).

Remarks. (1) $\mathcal{D}_\Psi$ defined above is chain homotopic to

$$\mathcal{D}_{(\delta\chi,\chi)} = \begin{pmatrix} (1+T)\delta\chi_0 \\ (-)^{n+2-r}(1+T)\chi_0 g^* \end{pmatrix} \ :\ E^{n+2-r} \to \mathcal{C}(g)_r = E_r \oplus D_{r-1}$$
$$= \begin{pmatrix} (1+T)\delta\chi_0 \\ (-)^{n-r}(1+T)\delta\psi'_0 g'^* + f'T\psi_0 k^* \\ (-)^{n+1-r}(1+T)\delta\psi''_0 g''^* - f''\psi_0 k^* \\ \psi_0 f'^* g'^* + T\psi_0 f''^* g''^* + (-)^{n-r}(1-T)\psi_1 k^* \end{pmatrix}$$
$$:\ E^{n+2-r} \to E_r \oplus D'_{r-1} \oplus D''_{r-1} \oplus C_{r-2}\ .$$

A chain homotopy $\mathcal{D}_\Psi \simeq \mathcal{D}_{(\delta\chi,\chi)}$ is given by

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ (-)^r\psi_0 k^* \end{pmatrix} : E^{n+2-r} \to E_{r+1} \oplus D'_r \oplus D''_r \oplus C_{r-1}.$$

Therefore $\mathcal{D}_\Psi$ is a chain equivalence if and only if $(\delta\chi, \chi)$ is Poincaré.
(2) If (Γ, Ψ) is Poincaré then there is a chain equivalence $\bar{\mathcal{D}}_\Psi : \mathcal{C}(-g'')^{n+2-*} \to \mathcal{C}(g')$ defined by:

$$\begin{pmatrix} (1+T)\delta\chi_0 + (-)^{r-1}k\psi_0 k^* & -g''(1+T)\delta\psi''_0 + (-)^n k(1+T)\psi_0 f''^* \\ (-)^{n-r}(1+T)\delta\psi'_0 g'^* + f'(1+T)\psi_0 k^* & (-)^{n-r-1}f'(1+T)\psi_0 f''^* \end{pmatrix}$$
$$: \mathcal{C}(-g'')^{n+2-r} = E^{n+2-r} \oplus D''^{n+1-r} \to \mathcal{C}(g')_r = E_r \oplus D'_{r-1}.$$

(3) Given $(n+1)$-dimensional quadratic structures $(\delta\psi', -\psi) \in \mathcal{Z}_{n+1}(f' : C \to D')$, $(\delta\psi'', \psi) \in \mathcal{Z}_{n+1}(f'' : C \to D'')$ and an $(n+2)$-dimensional quadratic structure $(\delta\chi, \chi = \delta\psi' \cup_\psi \delta\psi'')$ on a chain map $g : D' \cup_C D'' \to E$, we obtain a triad

$$\Gamma : \begin{tikzcd} C \arrow[r, "f'"] \arrow[d, "f''"'] \arrow[dr, squiggly, "k"] & D' \arrow[d, "g'"] \\ D'' \arrow[r, "-g''"'] & E \end{tikzcd}$$

where $g' : D' \to E$, $g'' : D'' \to E$, $k : g'f' \simeq -g''f''$ are defined by

$$g = (\, g' \quad (-)^r k \quad g'' \,) : D'_r \oplus C_{r-1} \oplus D''_r \to E_r \ ,$$

and $(\delta\chi, \delta\psi', \delta\psi'', -\psi)$ is an $(n+2)$-dimensional quadratic structure on the triad Γ.

We now describe the union operation for quadratic structures. Suppose we have two adjoining triads in $\mathbb{A}$:

$$\Gamma' : \begin{tikzcd} C \arrow[r, "f'"] \arrow[d, "f"'] \arrow[dr, squiggly, "k'"] & D' \arrow[d, "g'"] \\ D \arrow[r, "-\delta f'"'] & E' \end{tikzcd} \qquad \Gamma'' : \begin{tikzcd} C \arrow[r, "f"] \arrow[d, "f''"'] \arrow[dr, squiggly, "k''"] & D \arrow[d, "\delta f''"] \\ D'' \arrow[r, "-g''"'] & E'' \end{tikzcd}$$

and have two adjoining $(n+2)$-dimensional quadratic structures on them:

$$\Psi' = (\delta\chi', \delta\psi', -\delta\psi, \psi) \in \mathcal{Z}_{n+2}\Gamma' \ , \quad \Psi'' = (\delta\chi'', \delta\psi, \delta\psi'', \psi) \in \mathcal{Z}_{n+2}\Gamma'' \ .$$

Definition. The *union* $\Psi' \cup_{(\delta\psi,\psi)} \Psi''$ of Ψ' and Ψ'' along $(\delta\psi, \psi)$ is the quadratic structure

$$\Psi = (\delta\chi' \cup_{(\delta\psi,\psi)} \delta\chi'', \delta\psi', \delta\psi'', \psi)$$

on the triad

$$\Gamma : \begin{tikzcd} C \arrow[r, "f'"] \arrow[d, "f''"'] \arrow[dr, squiggly, "k"] & D' \arrow[d, "i_{E'} g'"] \\ D'' \arrow[r, "-i_{E''} g''"'] & E \end{tikzcd}$$

where

- E is the algebraic mapping cone of the chain map

$$\begin{pmatrix} \delta f' \\ \delta f'' \end{pmatrix} \ :\ D \to E' \oplus E'' \ ,$$

- $i_{E'} : E' \to E$ and $i_{E'} : E' \to E$ are the inclusion maps,
- k is a chain homotopy defined by

$$k = \begin{pmatrix} k' \\ (-)^r f \\ k'' \end{pmatrix} : C_r \to E_{r+1} = E'_{r+1} \oplus D_r \oplus E''_{r+1}.$$

- $\delta\chi' \cup_{(\delta\psi,\psi)} \delta\chi'' \in (W_\% E)_{n+2}$ is defined by

$$\begin{array}{l} (\delta\chi' \cup_{(\delta\psi,\psi)} \delta\chi'')_s \\ = \begin{pmatrix} \delta\chi'_s & 0 & 0 \\ (-)^{n+1-r}\delta\psi_s\delta f'^* - f\psi_s k'^* & (-)^{n-r-s}T\delta\psi_{s+1} & 0 \\ (-)^r k''\psi_s k'^* & (-)^s\delta f''\delta\psi_s + (-)^{n+1-s}k''\psi_s f^* & \delta\chi''_s \end{pmatrix} \end{array}$$

$$: E^{n+2-s-r} = E'^{n+2-s-r} \oplus D^{n+1-s-r} \oplus E''^{n+2-s-r} \to E_r = E'_r \oplus D_{r-1} \oplus E''_r \ ,$$

and its T-dual is represented by the matrix

$$\begin{pmatrix} T\delta\chi'_s & (-)^{s+1}\delta f' T\delta\psi_s + (-)^{n-s}k'T\psi_s f^* & (-)^{r-1}k'T\psi_s k''^* \\ 0 & (-)^{n+1-r-s}\delta\psi_{s+1} & (-)^{n-r}T\delta\psi_s\delta f''^* + fT\psi_s k''^* \\ 0 & 0 & T\delta\chi''_s \end{pmatrix}$$

$$: E^{n+2-s-r} = E'^{n+2-s-r} \oplus D^{n+1-s-r} \oplus E''^{n+2-s-r} \to E_r = E'_r \oplus D_{r-1} \oplus E''_r \ .$$

If (Γ', Ψ') , (Γ'', Ψ'') are $(n+2)$-dimensional quadratic triads, then so is (Γ, Ψ). This will be called the *union* of (Γ', Ψ'), (Γ'', Ψ'') along the pair $(f : C \to D, (\delta\psi, \psi))$.

Proposition. *If Ψ' on Γ' and Ψ'' on Γ'' are both Poincaré then the union $\Psi = \Psi' \cup_{(\delta\psi,\psi)} \Psi''$ is also Poincaré.*

Proof: We show that the duality map $\mathcal{D}_\Psi : E^{n+2-*} \to \mathcal{C}(\Gamma)$ is a chain equivalence. First note that E^{n+2-*} has an algebraic mapping cone structure of a certain chain map $\Omega E''^{n+2-*} \to \mathcal{C}(\delta f')^{n+2-*}$, since the boundary map is expressed by the matrix

$$\begin{array}{c} \\ E'^{n+3-r} \\ D^{n+2-r} \\ E''^{n+3-r} \end{array} \begin{array}{c} \begin{array}{ccc} E'^{n+2-r} & D^{n+1-r} & E''^{n+2-r} \end{array} \\ \begin{pmatrix} (-)^r d^* & 0 & 0 \\ (-)^n\delta f'^* & (-)^r d^* & (-)^n\delta f''^* \\ 0 & 0 & (-)^r d^* \end{pmatrix} \end{array}$$

$$:\ \mathcal{C}(\delta f')^{n+2-r} \oplus E''^{n+2-r} \to \mathcal{C}(\delta f')^{n+3-r} \oplus E''^{n+3-r} \ .$$

Also note that $\mathcal{C}(\Gamma)$ has an algebraic mapping cone structure of a certain chain map $\Omega\mathcal{C}(\Gamma'') \to \mathcal{C}(g')$, since the boundary map is expressed by the matrix

$$\begin{array}{c} \\ E'_{r-1} \\ D'_{r-2} \\ E''_{r-1} \\ D_{r-2} \\ D''_{r-2} \\ C_{r-3} \end{array}
\begin{array}{c} \begin{array}{cccccc} E'_r & D'_{r-1} & E''_r & D_{r-1} & D''_{r-1} & C_{r-2} \end{array} \\
\begin{pmatrix} d & (-)^{r-1}g' & 0 & (-)^{r-1}\delta f' & 0 & k' \\ 0 & d & 0 & 0 & 0 & (-)^r f' \\ 0 & 0 & d & (-)^{r-1}\delta f'' & (-)^{r-1}g'' & k'' \\ 0 & 0 & 0 & d & 0 & (-)^r f \\ 0 & 0 & 0 & 0 & d & (-)^r f'' \\ 0 & 0 & 0 & 0 & 0 & d \end{pmatrix} \end{array}$$

$$: \ \mathcal{C}(g')_r \oplus \mathcal{C}(\Gamma'')_r \to \mathcal{C}(g')_{r-1} \oplus \mathcal{C}(\Gamma'')_{r-1} \ .$$

$\mathcal{D}_\Psi : E^{n+2-r} \to \mathcal{C}(\Gamma)_r$ is represented by a block matirx of the form

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \ \mathcal{C}(\delta f')^{n+2-r} \oplus E''^{n+2-r} \to \mathcal{C}(g')_r \oplus \mathcal{C}(\Gamma'')_r \ ,$$

where

$$\begin{aligned}
\alpha &= \begin{pmatrix} (1+T)\delta\chi'_0 + (-)^{r-1}k'\psi_0 k'^* & -\delta f' T\delta\psi_0 + (-)^n k'(1+T)\psi_0 f^* \\ (-)^{n-r}(1+T)\delta\psi'_0 g'^* + f'(1+T)\psi_0 k'^* & (-)^{n+1-r} f'(1+T)\psi_0 f^* \end{pmatrix} \\
&= \bar{\mathcal{D}}_{\Psi'} - \begin{pmatrix} 0 & -\delta f'\delta\psi_0 \\ 0 & 0 \end{pmatrix} \ : \ E'^{n+2-r} \oplus D^{n+1-r} \to E'_r \oplus D'_{r-1} \ , \\
\beta &= \begin{pmatrix} (-)^{r-1}k'(1+T)\psi_0 k''^* \\ f'(1+T)\psi_0 k''^* \end{pmatrix} \ : \ E''^{n+2-r} \to E'_r \oplus D'_{r-1} \ , \\
\gamma &= \begin{pmatrix} 0 & \delta f''\delta\psi_0 \\ (-)^{n+1-r}\delta\psi_0\delta f'^* & (-)^{n+1-r}(1-T)\delta\psi_1 + (-)^{n+1-r} f\psi_0 f^* \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\
&\quad : \ E'^{n+2-r} \oplus D^{n+1-r} \to E''_r \oplus D_{r-1} \oplus D''_{r-1} \oplus C_{r-2} \\
\delta &= \begin{pmatrix} (1+T)\delta\chi''_0 + (-)^{r-1}k''\psi_0 k''^* \\ (-)^{n-r}T\delta\psi_0\delta f''^* + f(1+T)\psi_0 k''^* \\ (-)^{n-r}(1+T)\delta\psi''_0 g''^* \\ -(1+T)\psi_0 f''^* g''^* \end{pmatrix} \\
&= \mathcal{D}_{\Psi''} - \begin{pmatrix} 0 \\ (-)^{n-r}\delta\psi_0\delta f''^* \\ 0 \\ 0 \end{pmatrix} \ : \ E''^{n+2-r} \to E''_r \oplus D_{r-1} \oplus D''_{r-1} \oplus C_{r-2} \ .
\end{aligned}$$

The matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & (-)^{r-1}\delta\psi_0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$: \ E'^{n+2-r} \oplus D^{n+1-r} \oplus E''^{n+2-r} \to E'_{r+1} \oplus D'_r \oplus E''_{r+1} \oplus D_r \oplus D''_r \oplus C_{r-1}$$

defines a chain homotopy between $\mathcal{D}_\Psi$ and the chain equivalence

$$\begin{pmatrix} \bar{\mathcal{D}}_{\Psi'} & \beta \\ O & \mathcal{D}_{\Psi''} \end{pmatrix} : \mathcal{C}(\delta f')^{n+2-r} \oplus E''^{n+2-r} \to \mathcal{C}(g')_r \oplus \mathcal{C}(\Gamma'')_r \ .$$

Therefore $\mathcal{D}_\Psi$ is also a chain equivalence and Ψ is Poincaré. □

Unions of pairs and unions of cobordisms are special cases of the above. For example take adjoining $(n+1)$-dimensional Poincaré cobordisms:

$$\begin{aligned} &(f', f) : C' \oplus C \to D', (\delta\psi', \psi' \oplus -\psi) \\ &(g, g'') : C \oplus C'' \to D'', (\delta\psi'', \psi \oplus -\psi'') \end{aligned}$$

These can be viewed as adjoining quadratic Poincaré triads:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr, rightsquigarrow, "0"] & C' \arrow[d, "f'"] \\ C \arrow[r, "-f"'] & D' \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr, rightsquigarrow, "0"] & C \arrow[d, "g"] \\ C'' \arrow[r, "-g''"'] & D'' \end{tikzcd}$$

$$(\delta\psi', \psi', -\psi, 0) \qquad\qquad (\delta\psi'', \psi, -\psi'', 0)$$

The union of cobordisms described in the previous section is equal to the pair associated with the union of these triads.

4. Homotopy equivalences

In this section we discuss homotopy equivalences of quadratic complexes and pairs. We fix an additive category $\mathbb{A}$ with involution and $\epsilon = \pm 1$ as before.

Definition. A *map* (*resp. homotopy equivalence*) of n-dimensional quadratic complexes in $\mathbb{A}$

$$f \ : \ (C, \psi) \to (C', \psi')$$

is a chain equivalence $f : C \to C'$ in $\mathbb{A}$ such that

$$f_\%(\psi) = \psi' \in \mathcal{Z}_n C' \ .$$

Proposition. *If $f : (C, \psi) \to (C'\psi')$ is a map of n-dimensional quadratic complexes in $\mathbb{A}$, then*

$$(\, f \quad 1\,) : C \oplus C' \to C', \quad (0, \psi \oplus -\psi')$$

is a cobordism between (C, ψ) and (C, ψ'). If (C, ψ) is Poincaré and f is a homotopy equivalence, then the cobordism is also Poincaré.

Proof: It is easy to check that $(0, \psi \oplus -\psi')$ defines an $(n+1)$-dimensional quadratic structure on $(\, f \quad 1\,)$. Its duality map $\mathcal{D} : C'^{n+1-*} \to \mathcal{C}((\, f \quad 1\,))$ is the composite of the following four chain maps

$$C'^{n+1-*} \xrightarrow{f^*} C^{n+1-*} \underset{\cong}{\longrightarrow} S(C^{n-*}) \xrightarrow{S\mathcal{D}_\psi} SC \xrightarrow{{}^t(\,0 \quad 1 \quad -f\,)} \mathcal{C}((\, f \quad 1\,)) \ .$$

The second map is the isomorphism given in section 2. If f is a chain equivalence, then the first map is also a chain equivalence. If ψ is Poincaré, then the third map is a chain equivalence. The fourth map is always a chain equivalence. □

Definition. A *map* of $(n+1)$-dimensional quadratic pairs in $\mathbb{A}$

$$(f : C \to D, (\delta\psi, \psi)) \to (f' : C' \to D', (\delta\psi', \psi'))$$

is a chain complex triad in $\mathbb{A}$

$$\Gamma_{g,h,k} \ : \ \begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, squiggly, "k"] & D \arrow[d, "h"] \\ C' \arrow[r, "f'"'] & D' \end{tikzcd}$$

such that $(g_\%, h_\%; k_\%)(\delta\psi) = (\delta\psi', \psi')$; *i.e.*

$$\begin{aligned} \psi'_s &= g\psi_s g^* \ : \ C'^{n-r-s} \to C'_r \\ \delta\psi'_s &= h\delta\psi_s h^* + (-)^{n+1}(k\psi_s f^* h^* + (-)^r f' g\psi_s k^* + (-)^{n-r} tT\psi_{s+1} k^*) \\ &\quad : \ D'^{n+1-s-r} \to D'_r \ . \end{aligned}$$

Such a map is a *homotopy equivalence* if the chain maps g and h are chain equivalences.

Proposition. *Let* $\Gamma_{g,h,k} : (f : C \to D, (\delta\psi, \psi)) \to (f' : C' \to D', (\delta\psi', \psi'))$ *be a homotopy equivalence of* $(n+1)$*-dimensional quadratic pairs in* $\mathbb{A}$. *If* $(\delta\psi, \psi)$ *is Poincaré, then so is* $(\delta\psi', \psi')$.

Proof: The duality map $\mathcal{D}_{(\delta\psi',\psi')}$ is given by the matrix

$$\begin{pmatrix} (1+T)\delta\psi'_0 \\ (-)^{n+1-r}(1+T)\psi'_0 f'^* \end{pmatrix} \ : \ D'^{n+1-r} \to \mathcal{C}(f')_r = D'_r \oplus C'_{r-1} \ ,$$

and there are identities

$$\begin{aligned} (1+T)\delta\psi'_0 &= h(1+T)\delta\psi_0 h^* + (-)^{n+1}\{k\psi_0 f^* h^* + kT\psi_0 g^* f'^* \\ &\quad + (-)^r(f' g\psi_0 k^* + hfT\psi_0 k^*) + (-)^{n+1-r}k(1-T)\psi_1 k^*\} \ , \\ (-)^{n+1-r}(1+T)\psi'_0 f'^* &= (-)^{n+1-r} g(1+T)\psi_0 g^* f'^* \ . \end{aligned}$$

This map is chain homotopic to the following composite of three chain equivalences

$$\hat{\mathcal{D}} \ : \ D'^{n+1-*} \xrightarrow[\simeq]{h^*} D^{n+1-*} \xrightarrow[\simeq]{\mathcal{D}_{(\delta\psi,\psi)}} \mathcal{C}(f) \xrightarrow[\simeq]{(g,h;k)} \mathcal{C}(f') \ ,$$

which can be expressed by the matirx

$$\begin{pmatrix} h(1+T)\delta\psi_0 h^* + (-)^{n+1}k(1+T)\psi_0 f^* h^* \\ (-)^{n+1-r} g(1+T)\psi_0 f^* h^* \end{pmatrix} \ : \ D'^{n+1-r} \to D'_r \oplus C'_{r-1} \ .$$

The matrix

$$\begin{pmatrix} (-)^{n-r}kT\psi_0 k^* \\ (-)^{n+1} g(1+T)\psi_0 k^* \end{pmatrix} \ : \ D'_{n+1-r} \to D'_{r+1} \oplus C'_r$$

defines a desired chain homotopy $\hat{\mathcal{D}} \simeq \mathcal{D}_{(\delta\psi',\psi')}$. □

Proposition. *Suppose that* $\Gamma_{g,h,k} : (f : C \to D, (\delta\psi, \psi)) \to (f' : C' \to D', (\delta\psi', \psi'))$ *is a homotopy equivalence of* $(n+1)$*-dimensional quadratic pairs in* $\mathbb{A}$ *and that* $(\delta\psi, \psi)$ *is Poincaré. Then there is an* $(n+2)$*-dimensional quadratic Poincaré triad*

$$\Gamma \ : \ \begin{tikzcd} C \oplus C' \arrow[r, "f \oplus f'"] \arrow[d, "(g \ \ 1)"'] \arrow[dr, squiggly, "(k \ \ 0)"] & D \oplus D' \arrow[d, "(h \ \ 1)"] \\ C' \arrow[r, "f'"'] & D' \end{tikzcd} \quad , \quad (0, \delta\psi \oplus -\delta\psi', 0, \psi \oplus -\psi')$$

Proof: We show that the duality map $\mathcal{D}$ for the triad is a chain equivalence. The rest of the proof is straightforward.

The bondary map for $\mathcal{C}(\Gamma)$ is expressed by the following matrix.

$$\begin{array}{c} \\ D'_{r-1} \\ D_{r-2} \\ D'_{r-2} \\ C'_{r-2} \\ C_{r-3} \\ C'_{r-3} \end{array} \begin{array}{c} \begin{array}{cccccc} D'_r & D_{r-1} & D'_{r-1} & C'_{r-1} & C_{r-2} & C'_{r-2} \end{array} \\ \begin{pmatrix} d_{D'} & (-)^{r-1}h & (-)^{r-1} & (-)^r f' & k & 0 \\ 0 & d_D & 0 & 0 & (-)^r f & 0 \\ 0 & 0 & d_{D'} & 0 & 0 & (-)^r f' \\ 0 & 0 & 0 & d_{C'} & (-)^r g & (-)^r \\ 0 & 0 & 0 & 0 & d_C & 0 \\ 0 & 0 & 0 & 0 & 0 & d_{C'} \end{pmatrix} \end{array}$$

Therefore, we have a chain equivalence $p : \mathcal{C}(\Gamma) \to S\mathcal{C}(f')$;

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} : D'_r \oplus D_{r-1} \oplus D'_{r-1} \oplus C'_{r-1} \oplus C_{r-2} \oplus C'_{r-2} \to D'_{r-1} \oplus C'_{r-2} .$$

The duality map $\mathcal{D}$ is expressed by the following matrix.

$$\begin{array}{c} \\ D'_r \\ D_{r-1} \\ D'_{r-1} \\ C'_{r-1} \\ C_{r-2} \\ C'_{r-2} \end{array} \begin{array}{c} D'^{n+2-r} \\ \begin{pmatrix} (-)^{r-1} k\psi_0 k^* \\ (-)^{n-r}(1+T)\delta\psi_0 h^* + f(1+T)\psi_0 k^* \\ (-)^{n+1-r}(1+T)\delta\psi'_0 \\ 0 \\ (1+T)\psi_0 g^* f'^* \\ -(1+T)\psi'_0 f'^* \end{pmatrix} \end{array}$$

Therefore the following diagram commutes.

$$\begin{tikzcd} D'^{n+2-*} \arrow[r, "\mathcal{D}"] \arrow[d, "\cong"] & \mathcal{C}(\Gamma) \arrow[d, "p", "\simeq"'] \\ S(D'^{n+1-*}) \arrow[r, "\simeq", "\mathcal{D}_{(\delta\psi',\psi')}"'] & S\mathcal{C}(f') \end{tikzcd}$$

where the left vertical arrow is the isomorphism given in section 2. Therefore $\mathcal{D}$ is a chain equivalence. $\square$

5. Algebraic surgery

In this section we review the algebraic surgery on quadratic complexes.

6. Splitting for quadratic Poincaré pairs

In this section we describe a method to split Poincaré pairs into two Poincaré triads. As before we fix an additive category $\mathbb{A}$ with involution and $\epsilon = \pm 1$.

Let $c = (f : C \to D, (\delta\psi, \psi))$ be an $(n+1)$-dimensional quadratic Poincaré pair in $\mathbb{A}$, and C', D' be subcomplexes of C,D respectively. Recall that we are fixing identification

$$\begin{aligned} C\ &= \mathcal{C}(\rho_C : \Omega(C/C') \to C'), \\ D\ &= \mathcal{C}(\rho_D : \Omega(D/D') \to D'), \end{aligned}$$

for some suitable chain maps ρ_C and ρ_D and that we have the following inclusion maps and projection maps:

$$\begin{aligned} C'_r \ \underset{q_C}{\overset{i_C}{\rightleftarrows}}\ C_r = C'_r \oplus (C/C')_r \ \underset{j_C}{\overset{p_C}{\rightleftarrows}}\ (C/C')_r, \\ D'_r \ \underset{q_D}{\overset{i_D}{\rightleftarrows}}\ D_r = D'_r \oplus (D/D')_r \ \underset{j_D}{\overset{p_D}{\rightleftarrows}}\ (D/D')_r. \end{aligned}$$

Note that j_C, q_C, j_D, and q_D may not be chain maps in general and that we have the following identities:

$$\begin{aligned} \rho_C = (-)^r q_C d_C j_C : (C/C')_{r+1} \to C'_r \ , \\ \rho_D = (-)^r q_D d_D j_D : (D/D')_{r+1} \to D'_r \ . \end{aligned}$$

We assume that $p_D f i_C = 0$, *i.e.* $f : C \to D$ can be expressed by the matrix:

$$\begin{pmatrix} f' & \xi \\ 0 & \bar{f} \end{pmatrix} \ : \ C_r = C'_r \oplus (C/C')_r \longrightarrow D'_r \oplus (D/D')_r = D_r \ .$$

The two maps $f' : C' \to D'$ and $\bar{f} : C/C' \to D/D'$ are chain maps, but in general $\xi = q_D f j_C : C/C' \to D'$ is not a chain map. There is an identity

$$d_{D'}\xi + (-)^{r-1}\rho_D \bar{f} = (-)^{r-1} f'\rho_C + \xi d_{C/C'} \ : \ (C/C')_r \to D'_{r-1}$$

for each r.

We define an n-dimensional chain complex C'' by $(C/C')^{n-*}$, and an $(n+1)$-dimensional chain complex D'' by $\mathcal{C}(\bar{f})^{n+1-*}$. The boundary map for D'' is given by the matrix:

$$\begin{pmatrix} (-)^r d^*_{D/D'} & 0 \\ (-)^{n+1}\bar{f}^* & (-)^r d^*_{C/C'} \end{pmatrix} \;:\; D''_r = \mathcal{C}(\bar{f})^{n+1-r} = (D/D')^{n+1-r} \oplus (C/C')^{n-r}$$
$$\longrightarrow D''_{r-1} = \mathcal{C}(\bar{f})^{n+2-r} = (D/D')^{n+2-r} \oplus (C/C')^{n+1-r},$$

and, therefore, $C'' = (C/C')^{n-*}$ is a subcomplex of D''. We denote the inclusion map by $f'' : C'' \to D''$.

Recall that the duality maps $\mathcal{D}_\psi : C^{n-*} \to C$ and $\bar{\mathcal{D}}_{(\delta\psi,\psi)} : \mathcal{C}(f)^{n+1-*} \to D$ are defined by

$$\mathcal{D}_\psi = (1+T)\psi_0 \;:\; C^{n-r} \longrightarrow C_r,$$
$$\bar{\mathcal{D}}_{(\delta\psi,\psi)} = ((1+T)\delta\psi_0,\ f(1+T)\psi_0) \;:\; \mathcal{C}(f)^{n+1-r} = D^{n+1-r} \oplus C^{n-r} \longrightarrow D_r.$$

The projections $p_C : C \to C/C'$ and $p_D : D \to D/D'$ induce a chain map $p : \mathcal{C}(f) \to \mathcal{C}(\bar{f})$:

$$p \;=\; p_D \oplus p_C \;:\; \mathcal{C}(f)_r = D_r \oplus C_{r-1} \longrightarrow (D/D')_r \oplus (C/C')_{r-1} = \mathcal{C}(\bar{f})_r.$$

Consider the following commutative diagram:

$$\begin{tikzcd} C' \arrow[r, "i_C"] \arrow[d, "f'"'] & C \arrow[d, "f"] & C'' \arrow[l, "\mathcal{D}_\psi p_C^*"'] \arrow[d, "f''"] \\ D' \arrow[r, "i_D"'] & D & D'' \arrow[l, "\bar{\mathcal{D}}_{(\delta\psi,\psi)} p^*"] \end{tikzcd}$$

and define chain complexes B and $C^!$ as the pull-backs of the two rows:

$$B \;=\; \Omega\mathcal{C}(\ (i_C,\ \mathcal{D}_\psi p_C^*) \;:\; C' \oplus C'' \to C\ ),$$
$$C^! \;=\; \Omega\mathcal{C}(\ (i_D,\ \bar{\mathcal{D}}_{(\delta\psi,\psi)} p^*) \;:\; D' \oplus D'' \to D\ ).$$

Then we have a chain map $g^! : B \to C^!$:

$$g^! \;=\; f' \oplus f \oplus f'' \;:\; B_r = C'_r \oplus C_{r+1} \oplus C''_r \longrightarrow C^!_r = D'_r \oplus D_{r+1} \oplus D''_r.$$

Define an n-dimensional quadratic structure $(\delta\bar{\psi}, -\bar{\psi})$ on $g^! : B \to C^!$ as follows:

$$\delta\bar{\psi}_s = \begin{cases} \begin{pmatrix} 0 & 0 & 0 \\ 0 & (-)^{n-r-s}T\delta\psi_{s-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} & (s>0) \\ \begin{pmatrix} 0 & q_D(1+T)\delta\psi_0 & (-)^{nr+r}\rho' \\ 0 & 0 & (-)^{nr+r+1}j' \\ 0 & 0 & 0 \end{pmatrix} & (s=0) \end{cases}$$
$$: C^{!\,n-r-s} = D'^{n-r-s} \oplus D^{n+1-r-s} \oplus D''^{n-r-s} \longrightarrow C^!_r = D'_r \oplus D_{r+1} \oplus D''_r,$$

$$\bar{\psi}_s = \begin{cases} \begin{pmatrix} 0 & 0 & 0 \\ 0 & (-)^{n-r-s-1}T\psi_{s-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} & (s>0) \\ \begin{pmatrix} 0 & q_C(1+T)\psi_0 & (-)^{nr}\rho_C \\ 0 & 0 & (-)^{nr+1}j_C \\ 0 & 0 & 0 \end{pmatrix} & (s=0) \end{cases}$$

$$: B^{n-1-r-s} = C'^{n-1-r-s} \oplus C^{n-r-s} \oplus C''^{n-1-r-s} \longrightarrow B_r = C'_r \oplus C_{r+1} \oplus C''_r,$$

where

$$\rho' = (\rho_D,\ \xi)\ :\ D''^{n-r} = (D/D')_{r+1} \oplus (C/C')_r \longrightarrow D'_r,$$

$$j' = (j_D,\ 0)\ :\ D''^{n-r} = (D/D')_{r+1} \oplus (C/C')_r \longrightarrow D_{r+1}.$$

Proposition. *The structure* $(\delta\bar{\psi}, -\bar{\psi})$ *on* $g^!$ *is Poincaré.*

Proof: We will show that the duality map $\mathcal{D}_{(\delta\bar{\psi},-\bar{\psi})} : (C^!)^{n-*} \to \mathcal{C}(g^!)$ is a chain equivalence.

The boundary map for $C^!$ is given by the matrix

$$\begin{pmatrix} d_D & (-)^{n-r+1}i_D & (-)^{n-r+1}(1+T)\delta\psi_0 p_D^* & (-)^{n-r+1}f(1+T)\psi_0 p_C^* \\ 0 & d'_D & 0 & 0 \\ 0 & 0 & (-)^{n-r+1}(d_{D/D'})^* & 0 \\ 0 & 0 & (-)^{n+1}\bar{f}^* & (-)^{n-r+1}(d_{C/C'})^* \end{pmatrix}$$

$$:\ C^!_{n-r+1} = D_{n-r+2} \oplus D'_{n-r+1} \oplus (D/D')^r \oplus (C/C')^{r-1}$$

$$\longrightarrow\ C^!_{n-r} = D_{n-r+1} \oplus D'_{n-r} \oplus (D/D')^{r+1} \oplus (C/C')^r\ ,$$

and hence the boundary map for $(C^!)^{n-*}$ is represented by the matrix

$$\begin{pmatrix} (-)^{n+1}d_{D/D'} & (-)^{n+1-r}\bar{f} & (-)^{n+1+nr}\epsilon p_D(1+T)\delta\psi_0 & 0 \\ 0 & (-)^{n+1}d_{C/C'} & (-)^{nr+r}\epsilon p_C(1+T)\psi_0 f^* & 0 \\ 0 & 0 & (-)^r d_D^* & 0 \\ 0 & 0 & (-)^{n+1}i_D^* & (-)^r d_{D'}^* \end{pmatrix}$$

$$:\ (C^!)^{n-r} = (D/D')_{r+1} \oplus (C/C')_r \oplus D^{n+1-r} \oplus D'^{n-r}$$

$$\longrightarrow\ (C^!)^{n-r+1} = (D/D')_r \oplus (C/C')_{r-1} \oplus D^{n+2-r} \oplus D'^{n+1-r}\ .$$

to be completed later ....

[**Old proof**] The following proof uses a different definiton of duality maps and needs to be rewritten.

We will show that the duality map $\mathcal{D}_{(\delta\bar{\psi},-\bar{\psi})} : \mathcal{C}(g^!)^{n-*} \to C^!$ is essentially built up of four chain equivalences:

$$0 : \mathcal{C}(\bar{\epsilon}) \to 0$$

$$\beta_D = {}^t(\, j_D \quad -\rho_D\,)\ :\ D/D' \to \mathcal{C}(i_D)$$

$$\beta_D^* = (\, j_D^* \quad -\rho_D^*\,)\ :\ \mathcal{C}(i_D)^{n+1-*} \to (D/D')^{n+1-*}$$

$$\beta_C^* = (\, j_C^* \quad -\rho_C^*\,)\ :\ \mathcal{C}(i_C)^{n-*} \to (C/C')^{n-*}$$

where $\bar\epsilon$ is an isomorphism of a certain chain complex to itself described below.

We first describe how to combine the first two chain equivalences. Define a chain map $\epsilon : (-)^{n+1} \bullet \Omega^2 \mathcal{C}(\bar f) \to (-)^n \bullet \Omega(C/C')$ by

$$\epsilon_r = (\,0 \quad (-)^{n-r}\,) \ : \ (D/D')_{r+2} \oplus (C/C')_{r+1} \to (C/C')_{r+1} \ .$$

The boundary map of $A = \mathcal{C}(\epsilon)$ is given by the matrix:

$$\begin{pmatrix} (-)^n d_{C/C'} & 0 & (-)^n \\ 0 & (-)^{n+1} d_{D/D'} & (-)^{n+1-r}\bar f \\ 0 & 0 & (-)^{n+1} d_{C/C'} \end{pmatrix} : \begin{array}{l} (C/C')_{r+1} \oplus (D/D')_{r+1} \oplus (C/C')_r \\ \to (C/C')_r \oplus (D/D')_r \oplus (C/C')_{r-1} \ . \end{array}$$

Therefore A can be identified with the chain complex

$$\Omega \mathcal{C}(\hat f : \mathcal{C}(\ \bar\epsilon \ : \ (-)^{n+1} \bullet \Omega(C/C') \to (-)^n \bullet \Omega(C/C')\ ) \to (-)^{n+1} \bullet (D/D'))\ ,$$

where

$$\begin{aligned} \bar\epsilon_r &= (-)^{n-r} \ : \ (C/C')_{r+1} \to (C/C')_{r+1}\ , \\ \hat f_r &= (\,0 \quad (-)^{n+1}\bar f\,) : (C/C')_{r+1} \oplus (C/C')_r \to (D/D')_r \ . \end{aligned}$$

Since $\bar\epsilon$ is an isomorphism, $\mathcal{C}(\bar\epsilon)$ is contractible. In fact,

$$\gamma = \begin{pmatrix} 0 & 0 \\ (-)^r \bar\epsilon_r & 0 \end{pmatrix} : (C/C')_{r+1} \oplus (C/C')_r \to (C/C')_{r+2} \oplus (C/C')_{r+1}$$

is a chain contraction of $\mathcal{C}(\bar\epsilon)$, and we have a triad

$$\begin{tikzcd} \mathcal{C}(\bar\epsilon) \arrow[rr, "\hat f"] \arrow[dd] \arrow[ddrr, squiggly, "k"] & & (-)^{n+1} \bullet (D/D') \arrow[d, "\cong"', "\delta"] \\ & & D/D' \arrow[d, "\beta_D"] \\ 0 \arrow[rr] & & \mathcal{C}(i_D) \end{tikzcd}$$

where $\delta_r = (-)^{(n+1)(r-1)} : (D/D')_r \to (D/D')_r$ and k is a chain homotopy : $\beta_D \delta \hat f \simeq 0$. The most obvious choice for k may be $-\beta_D \delta \hat f \gamma$, but we use the chain homotopy

$$k = \begin{pmatrix} (-)^{nr-r} f j_C & 0 \\ (-)^{nr-r-1} f' \rho_C & (-)^{nr}\xi \end{pmatrix} \ : \ (C/C')_{r+1} \oplus (C/C')_r \to D_{r+1} \oplus D'_r$$

instead. This triad induces a chain equivalence

$$(0, \beta_D\delta; k) = (\,\beta_D \delta \quad (-)^r k\,) \ : \ \mathcal{C}(\hat f)_r = (D/D')_r \oplus \mathcal{C}(\bar\epsilon)_{r-1} \to \mathcal{C}(i_D)_r$$

We denote the skew desuspension of this by $\mathcal{B}_A : A \to \Omega\mathcal{C}(i_D)$:

$$\mathcal{B}_A = (\, -\beta_D\delta \quad (-)^{r+1}k\,) \ : \ A_r = (D/D')_{r+1} \oplus \mathcal{C}(\bar{\epsilon})_r \to \Omega\mathcal{C}(i_D)_r \ .$$

We next describe how to combine the two chain equivalences β_C^* and β_D^*. Consider the following triad :

$$\begin{tikzcd}
\mathcal{C}(i_D)^{n+1-*} \arrow[rr, "F"] \arrow[d, "-\beta_D^*"'] \arrow[drr, squiggly, "h"] & & S\mathcal{C}(i_C)^{n-*} \arrow[d, "-S\beta_C^*"] \\
(D/D')^{n+1-*} \arrow[rr, "G"'] & & S(C/C')^{n-*}
\end{tikzcd}$$

where

$$\begin{aligned}
F_r &= (-)^{n-r}(f^* \oplus f'^*) \ : \ D^{n+1-r} \oplus D'^{n-r} \to C^{n+1-r} \oplus C'^{n-r} \ , \\
G_r &= (-)^{n+1-r}\bar{f}^* \ : \ (D/D')^{n+1-r} \to (C/C')^{n+1-r} \ , \\
h &= (\,0 \quad (-)^{r-1}\xi^*\,) : D^{n+1-r} \oplus D'^{n-r} \to (C/C')^{n-r} \ .
\end{aligned}$$

Since the two vertical maps are chain equivalences, this diagram induces a chain equivalence $\mathcal{C}(F) \to \mathcal{C}(G)$, and hence a chain equivalence

$$\mathcal{B}_{A'} : A' = \Omega^2\mathcal{C}(F) = \mathcal{C}(\Omega^2 F) \to \Omega^2\mathcal{C}(G) \ .$$

To combine $\mathcal{B}_A$ and $\mathcal{B}_{A'}$, we need to find suitable maps between the sources and the targets respectively.

The map $A' \to A$ between the sources is the chain map induced by the following triad:

$$\begin{tikzcd}
\Omega^2\mathcal{C}(i_D)^{n+1-*} \arrow[rr, "\Omega^2 F"] \arrow[d] \arrow[drr, squiggly, "0"] & & \Omega\mathcal{C}(i_C)^{n-*} \arrow[d] \\
(-)^{n+1} \bullet \Omega^2\mathcal{C}(\bar{f}) \arrow[rr, "\epsilon"'] & & (-)^n \bullet \Omega(C/C')
\end{tikzcd}$$

where the two vertical maps are defined by:

$$\begin{aligned}
&(-)^{(n+1)(r+1)+1}\begin{pmatrix} p_D(1+T)\delta\psi_0 & 0 \\ (-)^{n+1-r}p_C(1+T)\psi_0 f^* & 0 \end{pmatrix} \\
&\qquad\qquad : \ D^{n+1-r} \oplus D'^{n-r} \to (D/D')_r \oplus (C/C')_{r-1} \\
&(-)^{nr+1}\,(\,p_C(1+T)\psi_0 \quad 0\,) \ : \ C^{n+1-r} \oplus C'^{n-r} \to (C/C')_{r-1} \ .
\end{aligned}$$

A direct calculation shows that the algebraic mapping cone of this chain map is equal to $\mathcal{C}(g^!)^{n-*}$.

The map $\Omega^2\mathcal{C}(G) \to \Omega\mathcal{C}(i_D)$ between the targets is the composition:

$$\Omega^2\mathcal{C}(G) = \Omega\mathcal{C}(\bar{f})^{n+1-*} \xrightarrow{\Omega p^*} \Omega\mathcal{C}(f)^{n+1-*} \xrightarrow{\Omega\mathcal{D}_{(\delta\psi,\psi)}} \Omega D \xrightarrow{\Omega(\text{inclusion})} \Omega\mathcal{C}(i_D) \ ,$$

and its algebraic mapping cone is equal to $C^!$.

Now we can fit $\mathcal{B}_A$ and $\mathcal{B}_{A'}$ together using the following triad to obtain a chain equivalence from $\mathcal{C}(g^!)^{n-*} \to C^!$:

$$\begin{tikzcd}
A' \arrow[r] \arrow[d, "\mathcal{B}_{A'}"'] \arrow[dr, squiggly, "H"] & A \arrow[d, "\mathcal{B}_A"] \\
\Omega^2\mathcal{C}(G) \arrow[r] & \Omega\mathcal{C}(i_D)
\end{tikzcd}$$

where the chain homotopy H is defined by

$$\begin{pmatrix} -f(1+T)\psi_0 q_C^* & 0 & (1+T)\delta\psi_0 g_D^* & 0 \\ 0 & (-)^{r+1} f' q_C(1+T)\psi_0 & 0 & (-)^r q_D(1+T)\delta\psi_0 \end{pmatrix}$$
$$: \ C'^{n-r} \oplus C^{n-r-1} \oplus D'^{n-r-1} \oplus D^{n-r} \to D_{r+2} \oplus D'_{r+1} \ .$$

A direct calculation shows that this chain equivalence is in fact equal to the duality map $\mathcal{D}_{(\delta\bar{\psi},-\bar{\psi})} : \mathcal{C}(g^!)^{n-*} \to C^!$. Therefore $(\delta\bar{\psi}, -\bar{\psi})$ is Poincaré. $\square$

Remark. The actual calculation was done in the reverse order. We first wrote down the duality map and the boundary homomorphisms of $\mathcal{C}(g^!)^{n-*}$ and $C^!$ in the form of matrices. Then we observed how the duality map is built up of chain equivalences.

[End of the Old Proof]

Now we have two $(n+1)$-dimensional quadratic triads in $\mathbb{A}$:

$$\Gamma' : \begin{tikzcd}
B \arrow[r, "-g'"] \arrow[d, "g^!"'] \arrow[dr, squiggly, "0"] & C' \arrow[d, "f'"] \\
C^! \arrow[r, "-\delta g'"'] & D'
\end{tikzcd} \qquad \Gamma'' : \begin{tikzcd}
B \arrow[r, "g^!"] \arrow[d, "g''"'] \arrow[dr, squiggly, "0"] & C^! \arrow[d, "\delta g''"] \\
C'' \arrow[r, "f''"'] & D''
\end{tikzcd}$$

$$\Psi' = (0, 0, \delta\bar{\psi}, \bar{\psi}) \qquad \Psi'' = (0, -\delta\bar{\psi}, 0, \bar{\psi}) \ ,$$

where $g' : B \to C'$, $g'' : B \to C''$, $\delta g' : C^! \to D'$, and $\delta g'' : C^! \to D''$ are the projections.

Proposition. *The quadratic triads (Γ', Ψ') and (Γ'', Ψ'') are Poincaré.*

Glueing these together, we obrtain a quadratic Poincaré triad.

$$\begin{tikzcd} B \arrow[r, "-g'"] \arrow[d, "g''"'] \arrow[dr, squiggly, "k"] & C' \arrow[d, "i'f'"] \\ C'' \arrow[r, "i''f''"'] & \bar{D} \end{tikzcd}$$

where $\bar{D}$ is the algebraic mapping cone $\mathcal{C}(\begin{pmatrix} -\delta g' \\ \delta g'' \end{pmatrix} : C^! \to D' \oplus D'')$, $i' : D' \to \bar{D}$, $i'' : D'' \to \bar{D}$ are the inclusion maps, and k is the chain homotopy ${}^t(\,0 \quad (-)^r g! \quad 0\,) : B_r \to D'_r \oplus C^!_{r-1} \oplus D''_r$. It then induces a quadratic Poincaré pair

$$\begin{aligned} &(\ \bar{f} : \bar{C} = \mathcal{C}(\begin{pmatrix} -g' \\ g'' \end{pmatrix} : B \to C' \oplus C'') \to \bar{D}\ , (0 \cup_{\delta\bar{\psi}} 0, 0 \cup_{\bar{\psi}} 0)\ ) \\ &\bar{f} = f' \oplus g^! \oplus (-f'') : C'_r \oplus B_{r-1} \oplus C''_r \to D'_r \oplus C^!_{r-1} \oplus D''_r\ . \end{aligned}$$

Proposition. *This pair is homotopy equivalent to the original pair* $(f : C \to D, (\delta\psi, \psi))$.

Proof: The inclusion maps

$$\begin{aligned} \bar{\imath}_C &= {}^t(\,0 \quad 0 \quad 1 \quad 0 \quad 0\,) : C \to \bar{C} \\ \bar{\imath}_D &= {}^t(\,0 \quad 0 \quad 1 \quad 0 \quad 0\,) : D \to \bar{D} \end{aligned}$$

are chain equivalences and the commutative diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\bar{\imath}_C"'] & D \arrow[d, "\bar{\imath}_D"] \\ \bar{C} \arrow[r, "\bar{f}"'] & \bar{D} \end{tikzcd}$$

gives the desired homotopy equivalence. $\square$