The Union of Quadratic Complexes

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1. Q-groups

We refer the reader to [**Ranicki**] for the terms which are not defined here. Let A be a ring with involution. In [**Ranicki**] $\epsilon \in A$ is a fixed central unit such that $\bar{\epsilon} = \epsilon^{-1} \in A$. Throughout this note we assume that $\epsilon = 1$ and omit it from notation.

The *algebraic mapping cone* $\mathcal{C}(f)$ of an A-module chain map $f : C \to D$ is the A-module chain complex defined by

$$d_{\mathcal{C}(f)} = \begin{pmatrix} d_D & (-)^{r-1} f \\ 0 & d_C \end{pmatrix} : \mathcal{C}(f)_r = D_r \oplus C_{r-1} \to \mathcal{C}(f)_{r-1} = D_{r-1} \oplus C_{r-2}.$$

Given A-module chain complexes C, D let $C^t \otimes_A D$, $\mathrm{Hom}_A(C, D)$ be the $\mathbb{Z}$-module chain complexes defined by

$$d_{C^t \otimes_A D} : (C^t \otimes_A D)_n = \sum_{p+q=n} C^t_p \otimes_A D_q \to (C^t \otimes_A D)_{n-1};$$
$$x \otimes y \mapsto x \otimes d_D(y) + (-)^q d_C(x) \otimes y,$$

$$d_{\mathrm{Hom}_A(C,D)} : \mathrm{Hom}_A(C, D)_n = \sum_{q-p=n} \mathrm{Hom}_A(C_p, D_q) \to \mathrm{Hom}_A(C, D)_{n-1};$$
$$f \mapsto d_D f + (-)^q f d_C \quad (: C_{q-n+1} \to C_q).$$

When C,C', D, D' are A-module chain complexes, there is a $\mathbb{Z}$-module chain map $\tau : \mathrm{Hom}_A(C, D) \otimes_{\mathbb{Z}} \mathrm{Hom}_A(C', D') \to \mathrm{Hom}_A(C \otimes_{\mathbb{Z}} C', D \otimes_{\mathbb{Z}} D')$ defined by

$$\tau : \mathrm{Hom}_A(C, D)_m \otimes_{\mathbb{Z}} \mathrm{Hom}_A(C', D')_n \to \mathrm{Hom}_A(C \otimes_{\mathbb{Z}} C', D \otimes_{\mathbb{Z}} D')_{m+n}$$
$$f \otimes g \mapsto \{x \otimes y (\in C_r \otimes_{\mathbb{Z}} C'_s) \mapsto (-)^{(m-r)s} f(x) \otimes g(y)\}.$$

Let C^* be the A-module chain complex defined by

$$d_{C^*} = (d_C)^* : (C^*)_r = C^{-r} \to (C^*)_{r-1} = C^{-r+1},$$

and let C^{n-*} $(n \in \mathbb{Z})$ be the A-module chain complex defined by

$$d_{C^{n-*}} = (-)^r (d_C)^* : (C^{n-*})_r = C^{n-r} \to (C^{n-*})_{r-1} = C^{n-r+1}.$$

Given a finite-dimensional A-module chain complex C let the generator $T \in \mathbb{Z}_2$ act on $\mathrm{Hom}_A(C^*, C)$ by the involution

$$T : \mathrm{Hom}_A(C^p, C_q) \to \mathrm{Hom}_A(C^q, C_p) \quad ; \qquad f \mapsto (-)^{pq} f^*,$$

and define a $\mathbb{Z}$-module chain complex $W_\% C$ by

$$W_\% C = W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathrm{Hom}_A(C^*, C),$$

where W is the standard free $\mathbb{Z}[\mathbb{Z}_2]$-resolution of $\mathbb{Z}$

$$W : \ldots \to \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \to 0.$$

$(W_\% C)_n$ is isomorphic to $\sum_{s \geq 0} \mathrm{Hom}_A(C^*, C)_{n-s}$ and an n-chain of $W_\% C$ can be viewed as a collection

$$\psi = (\psi_s \in \mathrm{Hom}_A(C^{n-r-s}, C_r))_{s \geq 0}.$$

Its boundary is

$$(d\psi_s + (-)^r \psi_s d^* + (-)^{n-s-1}(\psi_{s+1} + (-)^{s+1} T\psi_{s+1}) : C^{n-r-s-1} \to C_r \quad (r \in \mathbb{Z}))_{s \geq 0}.$$

$W_\% C$ is a $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex by the action: $T\psi = (T\psi_s)$. In particular, if ψ is a cycle then so is $T\psi$. In general ψ and $T\psi$ do not represent the same homology class. The set of n-cycles of $W_\% C$ is denoted $\mathcal{Z}_n C$. The homology groups of $W_\% C$ are denoted $Q_*(C)$ and are called the *quadratic Q-groups* $Q_*(C)$ of C;

$$Q_n(C) = H_n(W_\% C) \qquad (n \in \mathbb{Z}).$$

A chain map of finite-dimensional A-module chain complexes

$$f : C \longrightarrow D$$

induces a $\mathbb{Z}[\mathbb{Z}_2]$-module chain map

$$\mathrm{Hom}_A(f^*, f) : \mathrm{Hom}_A(C^*, C) \to \mathrm{Hom}_A(D^*, D) \quad ; \quad \phi \mapsto f\phi f^*,$$

and hence also a $\mathbb{Z}[\mathbb{Z}_2]$-module chain map $f_\% : W_\% C \to W_\% D$ and a morphism in the quadratic Q-groups $f_\% : Q_n C \to Q_n D$;

$$\psi = (\psi_s) \mapsto f_\% \psi = (f \psi_s f^*).$$

An A-module chain homotopy $g : f \simeq f' : C \to D$ does not in general determine a $\mathbb{Z}[\mathbb{Z}_2]$-module chain homotopy

$$\mathrm{Hom}_A(f^*, f) \simeq \mathrm{Hom}_A(f'^*, f') : \mathrm{Hom}_A(C^*, C) \to \mathrm{Hom}_A(D^*, D).$$

Nevertheless the Q-group morphisms induced by A-module chain map f depend only on the chain homotopy class of f. The following notion is very important here.

A $\mathbb{Z}_2$-*isovariant chain map* of $\mathbb{Z}[\mathbb{Z}_2]$-module chain complexes

$$f : C \to D$$

is a 0-cycle of $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}_{\mathbb{Z}}(C, D))$, i.e., a collection of $\mathbb{Z}$-module morphisms

$$f = (f_s \in \sum_{r \in \mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(C_r, D_{r+s}))_{s \geq 0}$$

such that

$$\begin{aligned} d_D f_s + (-)^{s-1} f_s d_C + (-)^{s-1}(f_{s-1} + (-)^s T f_{s-1}) = 0 \\ : C_r \to D_{r+s-1} \qquad (s \geq 0, \quad f_{-1} = 0). \end{aligned}$$

Here an exceptional boundary map

$$d_{\mathrm{Hom}_{\mathbb{Z}}(C,D)} : \mathrm{Hom}_{\mathbb{Z}}(C, D)_n \to \mathrm{Hom}_{\mathbb{Z}}(C, D)_{n-1} \quad ; \quad f \mapsto d_D f + (-)^{n-1} f d_C$$

is used, and $T \in \mathbb{Z}_2$ acts on $\mathrm{Hom}_{\mathbb{Z}}(C, D)$ by the involution

$$T : \mathrm{Hom}_{\mathbb{Z}}(C, D) \to \mathrm{Hom}_{\mathbb{Z}}(C, D) \quad ; \quad h \mapsto T_D h T_C.$$

Thus $f_0 : C \to D$ is a $\mathbb{Z}$-module chain map, $f_1 : f_0 \simeq T f_0 : C \to D$ is a $\mathbb{Z}$-module chain homotopy, and f_2, f_3, ... are higher $\mathbb{Z}$-module chain homotopies. A $\mathbb{Z}_2$-*isovariant chain homotopy* of $\mathbb{Z}_2$-isovariant chain maps

$$g : f \simeq f' : C \to D$$

is a 1-chain of $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}_{\mathbb{Z}}(C, D))$ whose boundary is $f' - f$, i.e., a collection of $\mathbb{Z}$-module morphisms

$$g = (g_s \in \sum_{r \in \mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(C_r, D_{r+s+1}))_{s \geq 0}$$

such that

$$\begin{aligned} f'_s - f_s = d_D g_s + (-)^s g_s d_C + (-)^s (g_{s-1} + (-)^s T g_{s-1}) \\ : C_r \to D_{r+s} \qquad (s \geq 0, \quad g_{-1} = 0). \end{aligned}$$

Examples. (1) A $\mathbb{Z}_2$-isovariant chain map $f : C \to D$ with $f_s = 0$ $(s \geq 1)$ is the same as a $\mathbb{Z}[\mathbb{Z}_2]$-module chain map $f_0 : C \to D$, and a $\mathbb{Z}_2$-isovariant chain homotopy $g : f \simeq f' : C \to D$ of such chain maps with $g_s = 0$ is the same as a $\mathbb{Z}[\mathbb{Z}_2]$-module chain homotopy $g_0 : f_0 \simeq f_0' : C \to D$.
(2) An A-module chain homotopy

$$g : f \simeq f' : C \to D$$

determines the $\mathbb{Z}_2$-isovariant chain homotopy

$$\mathrm{Hom}_A(g^*, g) : \mathrm{Hom}_A(f^*, f) \simeq \mathrm{Hom}_A(f'^*, f') : \mathrm{Hom}_A(C^*, C) \longrightarrow \mathrm{Hom}_A(D^*, D)$$

defined by

$$\mathrm{Hom}_A(g^*, g)_s : \mathrm{Hom}_A(C^*, C)_r = \sum_{p+q=r} \mathrm{Hom}_A(C^p, C_q) \longrightarrow \mathrm{Hom}_A(D^*, D)_{r+s+1};$$

$$\phi \mapsto \begin{cases} g\phi f^* + (-)^q f'\phi g^* : D^{r+s+1-q} \to D_q & \text{if } s = 0 \\ (-)^{q+1} g\phi g^* : D^{r+s+1-q} \to D_q & \text{if } s = 1 \\ 0 : D^{r+s+1-q} \to D_q & \text{if } s \geq 2. \end{cases}$$

(3) If $f : C \to D$ is a $\mathbb{Z}_2$-isovariant chain map of $\mathbb{Z}[\mathbb{Z}_2]$-module chain complexes, then $Tf = \{Tf_s\} : C \to D$ is also a $\mathbb{Z}_2$-isovariant chain map. f and Tf are $\mathbb{Z}_2$-isovariantly chain homotopic; a $\mathbb{Z}_2$-isovariant chain homotopy $g : f \simeq Tf : C \to D$ is defined by

$$g_s = \begin{cases} f_{s+1} : C_r \to D_{r+s+1}, & \text{if } s \text{ is even} \\ 0 : C_r \to D_{r+s+1}, & \text{if } s \text{ is odd.} \end{cases}$$

(4) Let $f : C \to D$ be an A-module chain map of A-module chain complexes. There is a $\mathbb{Z}_2$-isovariant chain map

$$U : \mathcal{C}(\mathrm{Hom}_A(f^*, f)) \to \mathrm{Hom}_A(\mathcal{C}(f)^*, \mathcal{C}(f))$$

defined by, for $(\delta\phi, \phi) \in \mathcal{C}(\mathrm{Hom}_A(f^*, f))_n = \mathrm{Hom}_A(D^*, D)_n \oplus \mathrm{Hom}_A(C^*, C)_{n-1}$

$$U_0(\delta\phi, \phi) = \left(\begin{pmatrix} \delta\phi & 0 \\ (-)^{n-r}\phi f^* & 0 \end{pmatrix} : \mathcal{C}(f)^{n-r} = D^{n-r} \oplus C^{n-r-1} \to \mathcal{C}(f)_r = D_r \oplus C_{r-1} \right)_r$$

$$U_1(\delta\phi, \phi) = \left(\begin{pmatrix} 0 & 0 \\ 0 & (-)^{r-1}\phi \end{pmatrix} : \mathcal{C}(f)^{n+1-r} = D^{n+1-r} \oplus C^{n-r} \to \mathcal{C}(f)_r = D_r \oplus C_{r-1} \right)_r$$

$$U_s(\delta\phi, \phi) = 0 \qquad \text{for} \quad s \geq 2.$$

The importance of $\mathbb{Z}_2$-isovariant chain maps and chain homotopies lies in the following fact: *a $\mathbb{Z}_2$-isovariant chain map $f : C \to D$ of $\mathbb{Z}[\mathbb{Z}_2]$-module chain complexes induces a $\mathbb{Z}$-module chain map*

$$f_\% : W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} C \to W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} D;$$

$$\psi = (\psi_s \in C_{n-s})_{s\geq 0} \mapsto f_\%(\psi) = (\sum_{r\geq s} (-)^{(r-s)(n+1)} f_{r-s}(T^{r-s}\psi_r) \in D_{n-s})_{s\geq 0},$$

and a $\mathbb{Z}_2$-isovariant chain homotopy $g : f \simeq f' : C \to D$ determines a $\mathbb{Z}$-module chain homotopy

$$g_{\%} : f_{\%} \simeq f'_{\%} : W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} C \to W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} D$$

defined by

$$\begin{aligned} g_{\%} &: (W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} C)_n \to (W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} D)_{n+1}; \\ &\psi = (\psi_s \in C_{n-s})_{s \geq 0} \mapsto g_{\%}(\psi) = (\sum_{r \geq s} (-)^{(r-s)(n+1)} g_{r-s}(T^{r-s}\psi_r) \in D_{n-s+1})_{s \geq 0}. \end{aligned}$$

These are the evaluations at $f \otimes \psi$ and $(-)^n g \otimes \psi$ respectively of the chain map

$$\begin{aligned} P &: \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}_{\mathbb{Z}}(C, D)) \otimes_{\mathbb{Z}} (W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} C) \to W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} D; \\ &(h_r \in \mathrm{Hom}_{\mathbb{Z}}(C, D)_{m+r})_{r \geq 0} \otimes (\psi_s \in C_{n-s})_{s \geq 0} \\ &\quad \mapsto (\sum_{r \geq s} (-)^{(m-r+s)n+r-s} h_{r-s}(T^{r-s}\psi_r) \in D_{m+n-s})_{s \geq 0}, \end{aligned}$$

which is the composition

$$\begin{aligned} &\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}_{\mathbb{Z}}(C, D)) \otimes_{\mathbb{Z}} (W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} C) \\ &\quad \cong \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}_{\mathbb{Z}}(C, D)) \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W^*, C) \\ &\quad \xrightarrow{\tau} \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W \otimes_{\mathbb{Z}} W^*, \mathrm{Hom}_{\mathbb{Z}}(C, D) \otimes_{\mathbb{Z}} C) \\ &\quad \xrightarrow{\nabla^*} \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W^*, \mathrm{Hom}_{\mathbb{Z}}(C, D) \otimes_{\mathbb{Z}} C) \\ &\quad \xrightarrow{\mathrm{ev}_*} \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W^*, D) \\ &\quad \cong W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} D. \end{aligned}$$

Here the third and the fourth maps are the chain maps induced by the $\mathbb{Z}[\mathbb{Z}_2]$-module chain maps

$$\begin{aligned} \nabla &: W^* \to W \otimes_{\mathbb{Z}} W^*; \\ 1_s^*(\in W^s) &\mapsto \sum_{p \geq 0} (-)^p 1_p \otimes T^p 1_{s+p}^* (\in \sum_{p \geq 0} W_p \otimes_{\mathbb{Z}} W^{s+p}) \end{aligned}$$

and

$$\begin{aligned} \mathrm{ev} &: \mathrm{Hom}_{\mathbb{Z}}(C, D) \otimes_{\mathbb{Z}} C \to D; \\ h \otimes x(\in \mathrm{Hom}_{\mathbb{Z}}(C, D)_r \otimes_{\mathbb{Z}} C_{s-r}) &\mapsto (-)^{r(s-r)} h(x)(\in D_s). \end{aligned}$$

In the case of the $\mathbb{Z}[\mathbb{Z}_2]$-module chain map $\mathrm{Hom}_A(f^*, f)$, this construction produces the $\mathbb{Z}[\mathbb{Z}_2]$-module chain map $f_{\%} : W_{\%}C \to W_{\%}D$ defined before, and the $\mathbb{Z}_2$-isovariant chain homotopy $\mathrm{Hom}_A(g^*, g)$ discussed above produces a $\mathbb{Z}$-module chain homotopy between $f_\%$ and $f'_\%$. More explicitly, if $g : f \simeq f' : C \to D$ is a chain homotopy of A-module chain maps, then a $\mathbb{Z}$-module chain homotopy $g_\% : f_\% \simeq f'_\% : W_\% C \to W_\% D$ is given by

$$g_\% : (W_\% C)_n \longrightarrow (W_\% D)_{n+1};$$
$$(\psi_s)_{s\geq 0} \mapsto (g\psi_s f^* + (-)^r f'\psi_s g^* + (-)^{n-r} g(T\psi_{s+1})g^* : D^{n-s+1-r} \to D_r)_{s\geq 0}.$$

Note that $g_\%$ depends not only on g but also on f and f'.

The composite of the $\mathbb{Z}_2$-isovariant chain maps $f : C \to D$, $g : D \to E$ is the $\mathbb{Z}_2$-isovariant chain map $gf : C \to E$ defined by

$$(gf)_s = \sum_{r=0}^{s} g_r(T^r f_{s-r}) : C_t \to E_{s+t} \qquad (s \geq 0, \quad t \in \mathbb{Z}).$$

This is the image of $g \otimes f$ by the following composite of $\mathbb{Z}[\mathbb{Z}_2]$-module chain maps

$$\begin{aligned}
&\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}_{\mathbb{Z}}(D,E)) \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}_{\mathbb{Z}}(C,D)) \\
&\xrightarrow{\tau} \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W \otimes_{\mathbb{Z}} W, \mathrm{Hom}_{\mathbb{Z}}(D,E) \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(C,D)) \\
&\xrightarrow{\Delta^*} \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}_{\mathbb{Z}}(D,E) \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(C,D)) \\
&\xrightarrow{c_*} \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}_{\mathbb{Z}}(C,E)).
\end{aligned}$$

Here Δ^* and c_* are the chain maps induced by the $\mathbb{Z}[\mathbb{Z}_2]$-module chain maps

$$\Delta : W \to W \otimes_{\mathbb{Z}} W;$$
$$1_s(\in W) \mapsto \sum_{p=0}^{s} 1_p \otimes T^p 1_{s-p} (\in \sum_{p=0}^{s} W_p \otimes_{\mathbb{Z}} W_{s-p})$$

and

$$c : \mathrm{Hom}_{\mathbb{Z}}(D,E) \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(C,D) \to \mathrm{Hom}_{\mathbb{Z}}(C,E);$$
$$h \otimes k (\in \mathrm{Hom}_{\mathbb{Z}}(D,E)_r \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(C,D)_{s-r}) \mapsto (-)^{r(s-r)} hk (\in \mathrm{Hom}_{\mathbb{Z}}(C,E)_s).$$

If $f : C \to D$, $g : D \to E$ are $\mathbb{Z}_2$-isovariant chain maps, then $(gf)_\% = g_\% f_\% : C \to E$; in fact for $\psi = (\psi_s \in C_{n-s})$

$$\begin{aligned}
g_\%(f_\%(\psi))_s &= \sum_{u\geq s} (-)^{(u-s)(n+1)} g_{u-s}(T_D^{u-s}(f_\%\psi)_u) \\
&= \sum_{u\geq s} (-)^{(u-s)(n+1)} g_{u-s}(T_D^{u-s}(\sum_{r\geq u} (-)^{(r-u)(n+1)} f_{r-u}(T_C^{r-u}\psi_r))) \\
&= \sum_{r\geq s} \sum_{t=0}^{r-s} (-)^{(r-s)(n+1)} g_{r-s-t}(T_D^{r-s-t}(f_t(T_C^{r-s-t}(T_C^{r-s}\psi_r)))) \\
&= \sum_{r\geq s} (-)^{(r-s)(n+1)} \left(\sum_{t=0}^{r-s} g_{r-s-t}(T^{r-s-t} f_t) \right) (T_C^{r-s}\psi_r) \\
&= (gf)_\%(\psi)_s.
\end{aligned}$$

The *suspension* SC and *desuspension* ΩC of an A-module chain complex C are the A-module chain complexes defined by

$$\begin{cases} d_{SC} = d_C : (SC)_r = C_{r-1} \longrightarrow (SC)_{r-1} = C_{r-2} \\ d_{\Omega C} = d_C : (\Omega C)_r = C_{r+1} \longrightarrow (\Omega C)_{r+1} = C_r \end{cases}$$

Given a finite dimensional A-module chain complex C define the *suspension* and *desuspension* maps in the quadratic Q-groups

$$S : Q_n(C) \longrightarrow Q_{n+1}(SC) \quad ; \quad \psi \mapsto S\psi \qquad (n \in \mathbb{Z})$$
$$\Omega : Q_n(C) \longrightarrow Q_{n-1}(\Omega C) \quad ; \quad \psi \mapsto \Omega\psi \qquad (n \in \mathbb{Z})$$

by

$$(S\psi)_s = (-)^{n+1-r-s}\psi_{s+1} : (SC)^{n+1-r-s} = C^{n-r-s} \to (SC)_r = C_{r-1},$$
$$(\Omega\psi)_s = (-)^{n-r-s}\psi_{s-1} : (\Omega C)^{n-1-r-s} = C^{n-r-s} \to (\Omega C)_r = C_{r+1}.$$

These actually define chain maps

$$S : S(W_{\%}C) \longrightarrow W_{\%}SC,$$
$$\Omega : \Omega(W_{\%}C) \longrightarrow W_{\%}\Omega C.$$

Given a chain map $f : C \to D$ of finite dimensional A-module chain complexes define the $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex $W_{\%}f$ by

$$W_{\%}f = \mathcal{C}(f_{\%} : W_{\%}C \to W_{\%}D).$$

The set of n-cycles of $W_{\%}f$ is denoted $\mathcal{Z}_n f$. It is a $\mathbb{Z}[\mathbb{Z}_2]$-module. The homology groups of $W_{\%}f$ are denoted $Q_*(f)$ and are called the *relative quadratic Q-groups* of the chain map f. The boundary of an $(n+1)$-chain $(\delta\psi, \psi) \in (W_{\%}D)_{n+1} \oplus (W_{\%}C)_n$ of $W_{\%}f$ is

$$\Big((d(\delta\psi_s) + (-)^r(\delta\psi_s)d^* + (-)^{n-s}(\delta\psi_{s+1} + (-)^{s+1}T\delta\psi_{s+1}) + (-)^n f\psi_s f^*$$
$$: D^{n-r-s} \to D_r \quad (r \in \mathbb{Z}))_{s\geq 0},$$
$$(d\psi_s + (-)^r\psi_s d^* + (-)^{n-s-1}(\psi_{s+1} + (-)^{s+1}T\psi_{s+1})$$
$$: C^{n-r-s-1} \to C_r \quad (r \in \mathbb{Z}))_{s\geq 0}\Big).$$

We describe the algebraic Thom construction. First it is to be noted that $W_{\%}f$ is isomorphic to $W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathcal{C}(\mathrm{Hom}_A(f^*, f))$. An $(n+1)$-chain of this chain complex is a collection

$$\{(\delta\psi_s, \psi_s) \in \mathrm{Hom}_A(D^*, D)_{n-s+1} \oplus \mathrm{Hom}_A(C^*, C)_{n-s} \quad | \quad s \geq 0\}$$

and the isomorphism is given by sending $(\delta\psi_s, \psi_s)_s$ to $(\delta\psi_s, (-)^s\psi_s)_s$. The $\mathbb{Z}_2$-isovariant chain map U described above induces a chain map $U_\% : W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathcal{C}(\mathrm{Hom}_A(f^*, f)) \to W_\%\mathcal{C}(f)$. This composed with the isomorphism above produces a chain map $\widehat{U}_\% : W_\% f \to W_\%\mathcal{C}(f)$. If $(\delta\psi, \psi)$ is an $(n+1)$-chain of $W_\% f$ then

$$\widehat{U}_\%(\delta\psi, \psi)_s = \begin{pmatrix} \delta\psi_s & 0 \\ (-)^{n+1-r}\psi_s f^* & (-)^{n-r-s}T\psi_{s+1} \end{pmatrix}$$
$$: \mathcal{C}(f)^{n+1-r-s} = D^{n+1-r-s} \oplus C^{n-r-s} \to \mathcal{C}(f)_r = D_r \oplus C_{r-1}.$$

If one uses $V = TU$ instead of U one gets another chain map $\widehat{V}_\% : W_\% f \to W_\%\mathcal{C}(f)$ and

$$\widehat{V}_\%(\delta\psi, \psi)_s = \begin{pmatrix} \delta\psi_s & (-)^s f\psi_s \\ 0 & (-)^{n+1-r-s}T\psi_{s+1} \end{pmatrix}$$
$$: \mathcal{C}(f)^{n+1-r-s} = D^{n+1-r-s} \oplus C^{n-r-s} \to \mathcal{C}(f)_r = D_r \oplus C_{r-1}.$$

There is a chain homotopy $h : \widehat{U}_\% \simeq \widehat{V}_\%$ defined by

$$h(\delta\psi, \psi)_s = \begin{pmatrix} 0 & 0 \\ 0 & (-)^{r+s-1}\psi_s \end{pmatrix} : \mathcal{C}(f)^{n+2-r-s} \to \mathcal{C}(f)_r.$$

In particular if $(\delta\psi, \psi)$ is an $(n+1)$-cycle of $W_\% f$ then $\widehat{U}_\%(\delta\psi, \psi)$ and $\widehat{V}_\%(\delta\psi, \psi)$ represent the same homology class in $Q_{n+1}(\mathcal{C}(f))$.

Next we describe the union operation. Let $f' : C \to D'$, $f'' : C \to D''$ be chain maps of finite dimensional A-module chain complexes and suppose we are given two $(n+1)$-cycles $(\delta\psi', \psi) \in \mathcal{Z}_{n+1}f'$, $(\delta\psi'', -\psi) \in \mathcal{Z}_{n+1}f''$. Let ϕ denote the $(n+1)$-cycle $\widehat{U}_\%(\delta\psi', \psi) \oplus \widehat{U}_\%(\delta\psi'', -\psi) \in W_\%(\mathcal{C}(f') \oplus \mathcal{C}(f''))$. Then $(0, T\phi)$ is an $(n+2)$-cycle in $W_\% I$, where $I : \mathcal{C}(f') \oplus \mathcal{C}(f'') \to SC$ is a chain map defined by

$$I = (\, 0 \quad -1 \quad 1 \quad 0\,) : (\mathcal{C}(f') \oplus \mathcal{C}(f''))_r = D'_r \oplus C_{r-1} \oplus C_{r-1} \oplus D''_r \to (SC)_r = C_{r-1}.$$

Note that $IT\phi_s I^* = 0$. $\widehat{V}_\%(0, T\phi)$ is an $(n+2)$-cycle of $W_\%(\mathcal{C}(I))$. There is a chain equivalence $G : \Omega\mathcal{C}(I) \to \mathcal{C}(\begin{pmatrix} f' \\ f'' \end{pmatrix} : C \to D' \oplus D'')$ defined by :

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & f'' & 0 & 1 \end{pmatrix}$$
$$: (\Omega\mathcal{C}(I))_r = D'_r \oplus C_{r-1} \oplus C_r \oplus C_{r-1} \oplus D''_r \to D'_r \oplus C_{r-1} \oplus D''_r.$$

Let D denote the algebraic mapping cone $\mathcal{C}(\begin{pmatrix} f' \\ f'' \end{pmatrix})$ and define the *union* $\delta\psi' \cup_\psi \delta\psi''$ to be $G_\%(\Omega\widehat{V}_\%(0, T\phi)) \in \mathcal{Z}_{n+1}D$; i.e.,

$$(\delta\psi' \cup_\psi \delta\psi'')_s = \begin{pmatrix} \delta\psi'_s & 0 & 0 \\ (-)^{n+1-r}\psi_s f'^* & (-)^{n-r-s}T\psi_{s+1} & 0 \\ 0 & (-)^{s+1}f''\psi_s & \delta\psi''_s \end{pmatrix}$$
$$: D^{n+1-r-s} = D'^{n+1-r-s} \oplus C^{n-r-s} \oplus D''^{n+1-r-s} \to D_r = D'_r \oplus C_{r-1} \oplus D''_r.$$

If we switch the order and take the union of $(\delta\psi'', -\psi)$ and $(\delta\psi', -(-\psi))$, then after reordering the direct summands,

$$(\delta\psi'' \cup_{-\psi} \delta\psi')_s = \begin{pmatrix} \delta\psi'_s & (-)^s f'\psi_s & 0 \\ 0 & (-)^{n+1-r-s}T\psi_{s+1} & (-)^{n-r}\psi_s f''^* \\ 0 & 0 & \delta\psi''_s \end{pmatrix}$$
$$: D^{n+1-r-s} = D'^{n+1-r-s} \oplus C^{n-r-s} \oplus D''^{n+1-r-s} \to D_r = D'_r \oplus C_{r-1} \oplus D''_r.$$

This represents the same homology class as $\delta\psi' \cup_\psi \delta\psi''$, because the difference $\delta\psi'' \cup_{-\psi} \delta\psi' - \delta\psi' \cup_\psi \delta\psi''$ is the boundary of an $(n+2)$-chain $\theta \in W_\% D$ defined by:

$$\theta_s = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (-)^{r+s-1}\psi_s & 0 \\ 0 & 0 & 0 \end{pmatrix} : D^{n+2-r} = D'^{n+2-s-r} \oplus C^{n+1-s-r} \oplus D''^{n+2-s-r}$$
$$\to D_r = D'_r \oplus C_{r-1} \oplus D''_r.$$

Suspension and desuspension operations can be applied to elements of $W_\%(f : C \to D)$; if $(\delta\psi, \psi) \in \mathcal{Z}_n f$, then $(S\delta\psi, -S\psi) \in \mathcal{Z}_{n+1}(f : SC \to SD)$ and $(\Omega\delta\psi, -\Omega\psi) \in \mathcal{Z}_{n-1}(f : \Omega C \to \Omega D)$.

2. L-groups

Let A be as in the previous section.

An *n-dimensional quadratic complex over A* (C, ψ) is an n-dimensional A-module chain complex C together with an element ψ of $\mathcal{Z}_n C$. Such a complex is *Poincaré* if

$$(1+T)\psi_0 : C^{n-*} \longrightarrow C$$

is a chain homotopy equivalence.

A *map* (resp. *homotopy equivalence*) of n-dimensional quadratic complexes over A

$$f : (C, \psi) \longrightarrow (C', \psi')$$

is an A-module chain map (resp. chain equivalence) $f : C \to C'$ such that

$$f_\%(\psi) = \psi' \in (W_\% C')_n.$$

An *$(n+1)$-dimensional quadratic pair over A* $(f : C \to D, (\delta\psi, \psi))$ is a chain map $f : C \to D$ from an n-dimensional A-module chain complex C to an $(n+1)$-dimensional A-module chain complex D together with an element $(\delta\psi, \psi)$ of $\mathcal{Z}_{n+1} f$. Such a pair is *Poincaré* if the A-module chain map $D^{n+1-*} \to \mathcal{C}(f)$ defined by

$$\begin{pmatrix} (1+T)\delta\psi_0 \\ (-)^{n+1-r}(1+T)\psi_0 f^* \end{pmatrix} : D^{n+1-r} \to \mathcal{C}(f)_r = D_r \oplus C_{r-1}$$

is a chain equivalence.

Remarks. (1) This chain map is a chain equivalence if and only if the A-module chain map $\mathcal{C}(f)^{n+1-*} \to D$ defined by

$$((1+T)\psi_0 \quad f(1+T)\psi_0) : \mathcal{C}(f)^{n+1-r} = D^{n+1-r} \oplus C^{n-r} \to D_r$$

is a chain equivalence.
(2) If $(f : C \to D, (\delta\psi, \psi))$ is an $(n+1)$-dimensional quadratic complex (which is Poincaré) then its *boundary* (C, ψ) is an n-dimensional quadratic complex (which is Poincaré).

A *cobordism* of n-dimensional quadratic complexes over A (C, ψ), (C', ψ') is an $(n+1)$-dimensional quadratic pair over A

$$(f \quad f') : C \oplus C' \to D, (\delta\psi, \psi \oplus -\psi') \in W_{\%}((f \quad f'))$$

with boundary $(C \oplus C', \psi \oplus -\psi')$. Such a cobordism is *Poincaré* if the pair above is Poincaré.

Examples. (1) If (C, ψ) is an n-dimensional quadratic complex over A, $f : C \to D$ is an A-module chain map, and D is n-dimensional then

$$((f \quad 1) : C \oplus D \to D, (0, \psi \oplus -f_{\%}\psi))$$

is a cobordism of (C, ψ) and $(D, f_{\%}\psi)$. If f is a chain equivalence, this cobordism is Poincaré.
(2) Suppose (C, ψ) is an n-dimensional quadratic (Poincaré) complex and ψ' is an element of $\mathcal{Z}_n C$ such that $d_{W_{\%}C}\delta\psi = \psi' - \psi$ for some $\delta\psi$, then $((1 \quad 1) : C \oplus C \to C, ((-)^n \delta\psi, \psi \oplus -\psi'))$ is a (Poincaré) cobordism of (C, ψ) and (C, ψ').

Proposition 2.1. *Poincaré cobordism is an equivalence relation on the set of n-dimensional quadratic Poincaré complexes over A. The Poincaré cobordism classes define an abelian group, the n-dimensional quadratic L-group of A, $L_n(A) \quad (n \geq 0)$, with addition and inverses by*

$$(C, \psi) + (C', \psi') = (C \oplus C', \psi \oplus \psi'), \qquad -(C, \psi) = (C, -\psi).$$

The key ingredient of the proof is the fact that one can glue adjoining Poincaré cobordisms to obtain a Poincaré cobordism. This will be established in the next section, and we describe a special case of the glueing operation in this section.

Suppose $(f' : C \to D', (\delta\psi', \psi))$, $(f'' : C \to D'', (\delta\psi'', -\psi))$ are $(n+1)$-dimensional quadratic pairs. The *union* of these is the $(n+1)$-dimensional quadratic complex

$(\mathcal{C}(\begin{pmatrix} f' \\ f'' \end{pmatrix}), \delta\psi' \cup_\psi \delta\psi'')$. We shall show that, if both pairs are Poincaré, then their union is also Poincaré. Let D denote $\mathcal{C}(\begin{pmatrix} f' \\ f'' \end{pmatrix})$ as before. D can be viewed as the algebraic mapping cone of the chain map $\Omega\mathcal{C}(f'') \to D'$ defined by

$$\begin{pmatrix} f' & 0 \end{pmatrix} : (\Omega\mathcal{C}(f''))_r = C_r \oplus D''_{r+1} \to D'_r$$

and D^{n+1-*} can be viewed as the algebraic mapping cone of the chain map $\Omega D''^{n+1-*} \to \mathcal{C}(f')^{n+1-*}$ defined by

$$\begin{pmatrix} 0 \\ (-)^{n+1-r} f''^* \end{pmatrix} : (\Omega D''^{n+1-*})_r = D''^{n-r} \to (\mathcal{C}(f')^{n+1-*})_r = D'^{n+1-r} \oplus C^{n-r}.$$

The following diagram commutes:

$$\begin{array}{ccc} \Omega D''^{n+1-*} & \xrightarrow{\begin{pmatrix} 0 \\ (-)^{n+1-r} f''^* \end{pmatrix}} & \mathcal{C}(f')^{n+1-*} \\ {\scriptstyle \begin{pmatrix} (-)^{n-r-1}(1+T)\psi_0 f''^* \\ (1+T)\delta\psi''_0 \end{pmatrix}} \Big\downarrow & & \Big\downarrow {\scriptstyle \begin{pmatrix} (1+T)\delta\psi'_0 & f'(1+T)\psi_0 \end{pmatrix}} \\ \Omega\mathcal{C}(f'') & \xrightarrow[\begin{pmatrix} f' & 0 \end{pmatrix}]{} & D'_r \end{array}$$

where vertical arrows are chain equivalences by assumption. Therefore we have a chain equivalence $D^{n+1-*} \to D$ defined by

$$\begin{pmatrix} (1+T)\delta\psi_0 & f'(1+T)\psi_0 & 0 \\ 0 & 0 & (-)^{n-r}(1+T)\psi_0 f''^* \\ 0 & 0 & (1+T)\delta\psi''_0 \end{pmatrix}$$
$$: D^{n+1-r} = D'^{n+1-r} \oplus C^{n-r} \oplus D''^{n+1-r} \to D_r = D'_r \oplus C_{r-1} \oplus D''_r,$$

which is chain homotopic to $(1+T)(\delta\psi' \cup_\psi \delta\psi'')_0$. A chain homotopy is given by

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & (-)^{r+1}\psi_0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : D'^{n+1-r} \oplus C^{n-r} \oplus D''^{n+1-r} \to D'_{r+1} \oplus C_r \oplus D''_{r+1}.$$

Therefore the union is Poincaré.

We can switch the order of the pairs. The two unions $(\mathcal{C}(\begin{pmatrix} f' \\ f'' \end{pmatrix}), \delta\psi' \cup_\psi \delta\psi'')$ and $(\mathcal{C}(\begin{pmatrix} f'' \\ f' \end{pmatrix}), \delta\psi'' \cup_{-\psi} \delta\psi')$ are cobordant. If both pairs are Poincaré, then these unions are Poincaré cobordant.

3. Triads

A *triad* Γ of A-module chain complexes

$$\begin{array}{ccc} C & \xrightarrow{f'} & D' \\ {\scriptstyle f''}\big\downarrow & \searrow^{k} & \big\downarrow{\scriptstyle g'} \\ D'' & \xrightarrow[g'']{} & E \end{array}$$

consists of A-module chain maps

$$f' : C \to D', \qquad f'' : C \to D'', \qquad g' : D' \to E, \qquad g'' : D'' \to E$$

and an A-module chain homotopy

$$k : g'f' \simeq g''f'' : C \to E.$$

Such a triad Γ induces an A-module chain map $(f'', g'; k) : \mathcal{C}(f') \to \mathcal{C}(g'')$;

$$(f'', g'; k) = \begin{pmatrix} g' & (-)^r k \\ 0 & f'' \end{pmatrix} : \mathcal{C}(f')_r = D'_r \oplus C_{r-1} \to \mathcal{C}(g'')_r = E_r \oplus D''_{r-1}.$$

The triad Γ induces a triad of $\mathbb{Z}[\mathbb{Z}_2]$-module chain complexes

$$\begin{array}{ccc} W_\% C & \xrightarrow{f'_\%} & W_\% D' \\ {\scriptstyle f''_\%}\big\downarrow & \searrow^{k_\%} & \big\downarrow{\scriptstyle g'_\%} \\ W_\% D'' & \xrightarrow[g''_\%]{} & W_\% E \end{array}$$

in which $f'_\%$,$f''_\%$,$g'_\%$,$g''_\%$ are $\mathbb{Z}[\mathbb{Z}_2]$-module chain maps but $k_\% : g'_\% f'_\% \simeq g''_\% f''_\%$ is only a $\mathbb{Z}$-module chain homotopy. $W_\%\Gamma$ will denote the $\mathbb{Z}$-module chain complex $\mathcal{C}((f''_\%, g'_\%; k_\%) : W_\% f' \to W_\% g'')$. A typical $(n+2)$ chain is a quadruple $(\delta\chi, \delta\psi', \delta\psi'', \psi) \in (W_\% E)_{n+2} \oplus (W_\% D')_{n+1} \oplus (W_\% D'')_{n+1} \oplus (W_\% C)_n$, and its boundary is:

$$\begin{aligned}
(d\delta\chi_s &+ (-)^r \delta\chi_s d^* + (-)^{n-s+1}(\delta\chi_{s+1} + (-)^{s+1} T\delta\chi_{s+1}) + (-)^{n+1} g'\delta\psi'_s g'^* \\
&+ (-)^{n+1} g''\delta\psi''_s g''^* + k\psi_s f'^* g'^* + (-)^r g'' f'' \psi_s k^* + (-)^{n-r} k(T\psi_{s+1})k^* \\
&: E^{n+1-s-r} \to E_r \qquad (r \in \mathbb{Z}), \\
d\delta\psi'_s &+ (-)^r \delta\psi'_s d^* + (-)^{n-s}(\delta\psi'_{s+1} + (-)^{s+1} T\delta\psi'_{s+1}) + (-)^n f'\psi_s f'^* \\
&: D'^{n-s-r} \to D'_r \qquad (r \in \mathbb{Z}), \\
d\delta\psi''_s &+ (-)^r \delta\psi''_s d^* + (-)^{n-s}(\delta\psi''_{s+1} + (-)^{s+1} T\delta\psi''_{s+1}) - (-)^n f''\psi_s f''^* \\
&: D''^{n-s-r} \to D''_r \qquad (r \in \mathbb{Z}), \\
d\psi_s &+ (-)^r \psi_s d^* + (-)^{n-s-1}(\psi_{s+1} + (-)^{s+1} T\psi_{s+1}) \\
&: C^{n-s-r-1} \to C_r \qquad (r \in \mathbb{Z}))_{s \geq 0}.
\end{aligned}$$

$\mathcal{Z}_n\Gamma$ denotes the set of n-cycles of $W_\%\Gamma$. The homology groups of $W_\%\Gamma$ are denoted $Q_*(\Gamma)$ and are called the *quadratic Q-groups* of Γ.

If $(\delta\chi, \delta\psi', \delta\psi'', \psi) \in \mathcal{Z}_{n+2}\Gamma$, then $(\delta\psi', \psi) \in \mathcal{Z}_{n+1}f'$ and $(\delta\psi'', -\psi) \in \mathcal{Z}_{n+1}f''$. We can glue these. Let $D = \mathcal{C}(\begin{pmatrix} f' \\ f'' \end{pmatrix})$ and $\chi = \delta\psi' \cup_\psi \delta\psi'' \in \mathcal{Z}_{n+1}D$. Define an A-module chain map $f : D \to E$ by

$$f = (\, g' \quad (-)^r k \quad -g'' \,) : D_r = D'_r \oplus C_{r-1} \oplus D''_r \to E_r,$$

then $(\delta\chi, \chi)$ is an element of $\mathcal{Z}_{n+2}f$. $\mathcal{C}(f)$ will be denoted $\mathcal{C}(\Gamma)$ and the *homology* A-modules $H_*(\Gamma)$ of Γ are defined by

$$H_n(\Gamma) = H_n(\mathcal{C}(\Gamma)) \qquad (n \in Z).$$

The boundary map of $\mathcal{C}(\Gamma)$ is:

$$d_{\mathcal{C}(\Gamma)} = \begin{pmatrix} d_E & (-)^{r-1}g' & (-)^r g'' & k \\ 0 & d_{D'} & 0 & (-)^r f' \\ 0 & 0 & d_{D''} & (-)^r f'' \\ 0 & 0 & 0 & d_C \end{pmatrix}$$
$$: \mathcal{C}(\Gamma)_r = E_r \oplus D'_{r-1} \oplus D''_{r-1} \oplus C_{r-2} \to \mathcal{C}(\Gamma)_{r-1} = E_{r-1} \oplus D'_{r-2} \oplus D''_{r-2} \oplus C_{r-3}.$$

An $(n+2)$-*dimensional quadratic triad over* A (Γ, Ψ) is a triad Γ of finite dimensional A-module chain complexes as above such that C is n-dimensional, D' and D'' are $(n+1)$-dimensional, E is $(n+2)$-dimensional, together with an element

$$\Psi = (\delta\chi, \delta\psi', \delta\psi'', \psi) \in \mathcal{Z}_{n+2}\Gamma.$$

Such a triad is *Poincaré* if

i) the $(n+1)$-dimensional quadratic pairs over A

$$(f' : C \to D', (\delta\psi', \psi) \in \mathcal{Z}_{n+1}f'),$$
$$(f'' : C \to D'', (\delta\psi'', -\psi) \in \mathcal{Z}_{n+1}f'')$$

are Poincaré, and

ii) the A-module chain map $(1+T)\Psi_0 : E^{n+2-*} \to \mathcal{C}(\Gamma)$ defined by

$$(1+T)\Psi_0 = \begin{pmatrix} (1+T)\delta\chi_0 + (-)^{r-1}k\psi_0 k^* \\ (-)^{n+2-r}(1+T)\delta\psi'_0 g'^* + f'(1+T)\psi_0 k^* \\ (-)^{n+1-r}(1+T)\delta\psi''_0 g''^* \\ (1+T)\psi_0 f''^* g''^* \end{pmatrix}$$
$$: E^{n+2-r} \to \mathcal{C}(\Gamma)_r = E_r \oplus D'_{r-1} \oplus D''_{r-1} \oplus C_{r-2}$$

is a chain equivalence.

Remarks. (1) $(1+T)\Psi_0$ defined above is chain homotopic to

$$\begin{pmatrix} (1+T)\delta\chi_0 \\ (-)^{n+2-r}(1+T)\chi_0 f^* \end{pmatrix} : E^{n+2-r} \to \mathcal{C}(f)_r = E_r \oplus D_{r-1}.$$

A chain homotopy is given by

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ (-)^r \psi_0 k^* \end{pmatrix} : E^{n+2-r} \to E_{r+1} \oplus D'_r \oplus D''_r \oplus C_{r-1}.$$

(2) If (Γ, Ψ) is Poincaré then there is a chain equivalence $\mathcal{C}(-g'')^{n+2-*} \to \mathcal{C}(g')$ which is defined by:

$$\begin{pmatrix} (1+T)\delta\chi_0 + (-)^{r-1}k\psi_0 k^* & -g''(1+T)\delta\psi''_0 + (-)^n k(1+T)\psi_0 f''^* \\ (-)^{n-r}(1+T)\delta\psi'_0 g'^* + f'(1+T)\psi_0 k^* & (-)^{n-r-1} f'(1+T)\psi_0 f''^* \end{pmatrix}$$
$$: \mathcal{C}(-g'')^{n+2-r} = E^{n+2-r} \oplus D''^{n+1-r} \to \mathcal{C}(g')_r = E_r \oplus D'_{r-1}.$$

Next we describe the algebraic Thom construction for triads. Let Γ be as above. Then we have the following triad:

$$\begin{array}{ccc} W_\% f' & \xrightarrow{(f''_\%, g'_\%; k_\%)} & W_\% g'' \\ \Big\downarrow \widehat{U}_\% & \searrow \widehat{K}_\% & \Big\downarrow \widehat{U}_\% \\ W_\% \mathcal{C}(f') & \xrightarrow[(f'', g'; k)_\%]{} & W_\% \mathcal{C}(g'') \end{array}$$

where $\widehat{K}_\% : \widehat{U}_\%(f''_\%, g'_\%; k_\%) \simeq (f'', g'; k)_\% \widehat{U}_\%$ is a chain homotopy defined by

$$\widehat{K}_\% : (W_\% f')_n \to (W_\% \mathcal{C}(g''))_{n+1}$$
$$(\delta\psi, \psi) \mapsto (\begin{pmatrix} 0 & 0 \\ (-)^{n+1} f'' \psi_s k^* & 0 \end{pmatrix} : \mathcal{C}(g'')^{n+1-s-r} \to \mathcal{C}(g'')_r)_{s \geq 0}.$$

This triad induces a $\mathbb{Z}$-module chain map

$$\widehat{U}_\% : W_\% \Gamma \to W_\%(f'', g'; k)$$

which sends an n-chain $(\delta\chi, \delta\psi', \delta\psi'', \psi)$ of $W_\%\Gamma$ to $(\delta\phi, \phi) \in (W_\%\mathcal{C}(g''))_n \oplus (W_\%\mathcal{C}(f'))_{n-1}$, where

$$\delta\phi_s = \begin{pmatrix} \delta\chi_s & 0 \\ (-)^{n-r}\delta\psi''_s g''^* + f''\psi_s k^* & (-)^{n-r-s-1} T\delta\psi''_{s+1} \end{pmatrix}$$
$$: \mathcal{C}(g'')^{n-r-s} = E^{n-r-s} \oplus D''^{n-r-s-1} \to \mathcal{C}(g'')_r = E_r \oplus D''_{r-1},$$
$$\phi_s = \begin{pmatrix} \delta\psi'_s & 0 \\ (-)^{n-r-1}\psi_s f'^* & (-)^{n-r-s} T\psi_{s+1} \end{pmatrix}$$
$$: \mathcal{C}(f')^{n-r-s-1} = D'^{n-r-s-1} \oplus C^{n-r-s-2} \to \mathcal{C}(f')_r = D'_r \oplus C_{r-1}.$$

If we use $\widehat{V}_\%$ instead of $\widehat{U}_\%$ then we get a chain map, also denoted $\widehat{V}_\%$, which sends an n-chain $(\delta\chi, \delta\psi', \delta\psi'', \psi)$ of $W_\%\Gamma$ to $(\delta\phi, \phi) \in (W_\%\mathcal{C}(g'))_n \oplus (W_\%\mathcal{C}(f''))_{n-1}$, where

$$\begin{aligned}
\delta\phi_s &= \begin{pmatrix} \delta\chi_s + (-)^{r-1} k\psi_s k^* & (-)^s g''\delta\psi''_s + (-)^{n-1-s} k\psi_s f'^* \\ 0 & (-)^{n-r-s} T\delta\psi''_{s+1} \end{pmatrix} \\
&\quad : \mathcal{C}(g'')^{n-r-s} = E^{n-r-s} \oplus D''^{n-r-s-1} \to \mathcal{C}(g'')_r = E_r \oplus D''_{r-1}, \\
\phi_s &= \begin{pmatrix} \delta\psi'_s & (-)^s f'\psi_s \\ 0 & (-)^{n-r-s-1} T\psi_{s+1} \end{pmatrix} \\
&\quad : \mathcal{C}(f')^{n-r-s-1} = D'^{n-r-s-1} \oplus C^{n-r-s-2} \to \mathcal{C}(f')_r = D'_r \oplus C_{r-1}).
\end{aligned}$$

We describe the union operation for triads. Suppose we have two triads:

$$\Gamma' : \begin{array}{ccc} C & \xrightarrow{f'} & D' \\ {\scriptstyle f}\downarrow & \searrow^{k'} & \downarrow{\scriptstyle g'} \\ D & \xrightarrow[\delta f']{} & E' \end{array} \qquad \Gamma'' : \begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle f''}\downarrow & \searrow^{k''} & \downarrow{\scriptstyle -\delta f''} \\ D'' & \xrightarrow[g'']{} & E'' \end{array}$$

and two $(n+2)$-cycles $\Psi' = (\delta\chi', \delta\psi', \delta\psi, \psi) \in \mathcal{Z}_{n+2}\Gamma'$, $\Psi'' = (\delta\chi'', -\delta\psi, \delta\psi'', \psi) \in \mathcal{Z}_{n+2}\Gamma''$. Consider the following triad $\widehat{\Gamma}''$ and an $(n+2)$-cycle $\widehat{\Psi}'' = (\widehat{\delta\chi}'', \delta\psi'', -\delta\psi, -\psi) \in \mathcal{Z}_{n+2}\widehat{\Gamma}''$

$$\begin{array}{ccc} C & \xrightarrow{f''} & D'' \\ {\scriptstyle f}\downarrow & \searrow^{k''} & \downarrow{\scriptstyle -g''} \\ D & \xrightarrow[\delta f'']{} & E'' \end{array}$$

where $\widehat{\delta\chi}''_s = \delta\chi''_s + (-)^{r+1} k''\psi_s k''^* : E''^{n+2-r-s} \to E_r$. Define $(\delta\phi, \phi) \in \mathcal{Z}_{n+2}(f, g'; k) \oplus (f, -g''; k'')$ by $\widehat{U}_\%(\Psi') \oplus \widehat{U}_\%(\widehat{\Psi}'')$ and consider an $(n+3)$-chain $\Phi = (0, T\delta\phi, 0, -T\phi)$ of $W_\%\widehat{\Gamma}$, where $\widehat{\Gamma}$ is the following triad:

$$\widehat{\Gamma} : \begin{array}{ccc} \mathcal{C}(f') \oplus \mathcal{C}(f'') & \xrightarrow{I} & SC \\ {\scriptstyle (-(f,g';k'))\oplus}\downarrow{\scriptstyle (-(f,-g'';k''))} & \searrow^{0} & \downarrow{\scriptstyle -f} \\ \mathcal{C}(\delta f') \oplus \mathcal{C}(-\delta f'') & \xrightarrow[J]{} & SD \end{array}$$

I and J are chain maps defined by

$$\begin{aligned}
I &= \begin{pmatrix} 0 & -1 & 1 & 0 \end{pmatrix} : D'_r \oplus C_{r-1} \oplus C_{r-1} \oplus D''_r \to C_{r-1} \\
J &= \begin{pmatrix} 0 & -1 & 1 & 0 \end{pmatrix} : E'_r \oplus D_{r-1} \oplus D_{r-1} \oplus E''_r \to D_{r-1}.
\end{aligned}$$

A direct calculation shows that Φ is a cycle. Let $F : \mathcal{C}(I) \to \mathcal{C}(J)$ be the chain map induced from $\widehat{\Gamma}$, and define $(\delta\theta, \theta) \in \mathcal{Z}_{n+3}F \subset W_\%\mathcal{C}(J)_{n+3} \oplus W_\%\mathcal{C}(I)_{n+2}$ to be $\widehat{V}_\%(\Phi)$.

Then $(\Omega\delta\theta, -\Omega\theta)$ is an $(n+2)$-cycle of $W_{\%}(F : \Omega\mathcal{C}(I) \to \Omega\mathcal{C}(J))$. I and J were chosen so that $\Omega\mathcal{C}(I)$, $\Omega\mathcal{C}(J)$ are chain equivalent to $\mathcal{C}(\begin{pmatrix} f' \\ f'' \end{pmatrix})$, $\mathcal{C}(\begin{pmatrix} \delta f' \\ \delta f'' \end{pmatrix})$ respectively. Now there is a triad

$$\begin{array}{ccc} \Omega\mathcal{C}(I) & \xrightarrow{\quad F \quad} & \Omega\mathcal{C}(J) \\ \big\downarrow & \overset{H}{\searrow} & \big\downarrow \\ \mathcal{C}(\begin{pmatrix} f' \\ f'' \end{pmatrix}) & \xrightarrow[\widetilde{F}]{\qquad} & \mathcal{C}(\begin{pmatrix} \delta f' \\ \delta f'' \end{pmatrix}) \end{array}$$

where the vertical maps are chain equivalences

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & f'' & 0 & 1 \end{pmatrix} : D'_r \oplus C_{r-1} \oplus C_r \oplus C_{r-1} \oplus D''_r \to D'_r \oplus C_{r-1} \oplus D''_r,$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \delta f'' & 0 & 1 \end{pmatrix} : E'_r \oplus D_{r-1} \oplus D_r \oplus D_{r-1} \oplus E''_r \to E'_r \oplus D_{r-1} \oplus E''_r,$$

$\widetilde{F}$ is a chain map defined by

$$\widetilde{F} = \begin{pmatrix} -g' & (-)^{r-1}k' & 0 \\ 0 & -f & 0 \\ 0 & (-)^{r-1}k'' & g'' \end{pmatrix} : D'_r \oplus C_{r-1} \oplus D''_r \to E'_r \oplus D_{r-1} \oplus E''_r,$$

and H is the chain homotopy defined by

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k'' & 0 & 0 \end{pmatrix} : D'_r \oplus C_{r-1} \oplus C_r \oplus C_{r-1} \oplus D''_r \to E'_{r+1} \oplus D_r \oplus E''_{r+1}.$$

Define $(\delta\chi, \chi) \in \mathcal{Z}_{n+2}\widetilde{F}$ to be the image of $(\Omega\delta\theta, -\Omega\theta)$ via the chain map $W_{\%}F \to W_{\%}\widetilde{F}$ induced from this triad. Then obviously $\chi = \delta\psi' \cup_{\psi} \delta\psi''$. Let E denote the algebraic mapping cone $\mathcal{C}(\begin{pmatrix} \delta f' \\ \delta f'' \end{pmatrix})$ and define $\delta\chi' \cup_{\delta\psi} \delta\chi'' \in W_{\%}E$ by $\delta\chi$; i.e.,

$$(\delta\chi' \cup_{\delta\psi} \delta\chi'')_s = \begin{pmatrix} \delta\chi'_s & 0 & 0 \\ (-)^{n-r}\delta\psi_s\delta f'^* + f\psi_s k'^* & (-)^{n+1-r-s}T\delta\psi_{s+1} & 0 \\ (-)^r k''\psi_s k'^* & (-)^{s+1}\delta f''\delta\psi_s + (-)^{n-s}k''\psi_s f^* & \delta\chi''_s \end{pmatrix}$$
$$: E^{n+2-s-r} = E'^{n+2-s-r} \oplus D^{n+1-s-r} \oplus E''^{n+2-s-r} \to E_r = E'_r \oplus D_{r-1} \oplus E''_r.$$

$(\delta\chi, \chi) \in \mathcal{Z}_{n+2}\widetilde{F}$ is induced (in the way described above) from the following triad Γ and an $(n+2)$-cycle Ψ of $W_{\%}\Gamma$:

$$
\begin{array}{ccc}
C & \xrightarrow{\quad f' \quad} & D' \\
{\scriptstyle f''}\big\downarrow & \overset{k}{\searrow} & \big\downarrow {\scriptstyle \begin{pmatrix} g' \\ 0 \\ 0 \end{pmatrix}} \\
D'' & \xrightarrow[\,^t(0 \;\; 0 \;\; g'')]{} & E
\end{array}
$$

$$\Psi = (\delta\chi' \cup_{\delta\psi} \delta\chi'', \delta\psi', \delta\psi'', \psi)$$

where k is a chain homotopy defined by

$$k = \begin{pmatrix} k' \\ (-)^{r+1} f \\ k'' \end{pmatrix} : C_r \to E_{r+1} = E'_{r+1} \oplus D_r \oplus E''_{r+1}.$$

If (Γ', Ψ') , (Γ'', Ψ'') are $(n+2)$-dimensional quadratic triads, then so is (Γ, Ψ) constructed above. This will be called the *union* of (Γ', Ψ'), (Γ'', Ψ''). If (Γ', Ψ'), (Γ'', Ψ'') are both Poincaré then the union is also Poincaré. In fact $(1+T)\Psi_0$ is chain homotopic to a chain map of the form

$$\begin{pmatrix} A & * \\ O & (1+T)\Psi''_0 \end{pmatrix} : \mathcal{C}(\delta f')^{n+2-r} \oplus E''^{n+2-r} \to \mathcal{C}(-g')_r \oplus \mathcal{C}(\Gamma'')_r$$

where A is the chain equivalence

$$\begin{pmatrix} (1+T)\delta\chi'_0 + (-)^{r-1} k'\psi_0 k'^* & \delta f'(1+T)\delta\psi_0 + (-)^{n-1} k'(1+T)\psi_0 f^* \\ (-)^{n-r-1}(1+T)\delta\psi'_0 g'^* - f'(1+T)\psi_0 k'^* & (-)^{n-r-1} f'(1+T)\psi_0 f^* \end{pmatrix}$$
$$: \mathcal{C}(\delta f')^{n+2-r} = E'^{n+2-r} \oplus D^{n+1-r} \to \mathcal{C}(-g')_r = E'_r \oplus D'_{r-1}.$$

Therefore $(1+T)\Psi_0$ is a chain equivalence and (Γ, Ψ) is Poincaré.

Since a Poincaré cobordism of n-dimensional quadratic complexes can be viewed as an $(n+1)$-dimensional quadratic Poincaré triad, the union of adjoining Poincaré cobordisms

$$(f', f) : C' \oplus C \to D', (\delta\psi', \psi' \oplus -\psi)$$
$$(g', g) : C \oplus C'' \to D'', (\delta\psi'', \psi \oplus -\psi'')$$

can be obtained by glueing the following quadratic Poincaré triads:

$$
\begin{array}{ccc}
0 & \longrightarrow & C' \\
\big\downarrow & \overset{0}{\searrow} & \big\downarrow {\scriptstyle -f'} \\
C & \xrightarrow[f]{} & D'
\end{array}
\qquad (\delta\psi', \psi', -\psi, 0)
$$

$$\begin{array}{ccc} 0 & \longrightarrow & C \\ \big\downarrow & \overset{0}{\searrow} & \big\downarrow{\scriptstyle -g} \\ C'' & \xrightarrow[g'']{} & D'' \end{array} \qquad (\delta\psi'', \psi, -\psi'', 0).$$

This finishes the proof of 2.1.