On isovariant rigidity of $\mathrm{CAT}(0)$ manifolds, I

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Topological Rigidity, I

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Suppose M is a closed topological manifold. If $f : M \to X$ is a homotopy equivalence, then f is homotopic to a homeomorphism.

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Let X be an **aspherical**, *closed topological manifold.*
Suppose M is a closed topological manifold. If $f : M \to X$ is a homotopy equivalence, then f is homotopic to a homeomorphism.

We say that X is **topologically rigid** if this is true.

Topological Rigidity, II

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The case $n \leq 2$ is true by the classification of curves and surfaces.
The case $n \geq 5$ was proven by Hsiang–Wall (1969).
The case $n = 4$ was proven by Freedman–Quinn (1990).
The case $n = 3$ was by Waldhausen (1968) and Perelman (2003).

Topological Rigidity, III

That example is a member of a certain differential-geometric class.

Theorem (Farrell–Jones, 1993)

Let X be a closed smooth manifold of dimension ≥ 5.
Suppose X admits a Riemannian metric
such that each sectional curvature is non-positive (≤ 0).
Then X is topologically rigid.

Topological Rigidity, IV

That hypothesis can be weakened with geometric group theory.

Theorem (Bartels–Lück, 2009)

Let X be a closed topological manifold of dimension ≥ 5. Suppose the universal cover $\widetilde{X}$ admits a $\mathrm{CAT}(0)$ metric and the fundamental group $\pi_1(X)$ acts isometrically on $\widetilde{X}$. Then X is topologically rigid.

Equivariant Rigidity of the Torus, I

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Theorem (Connolly–Davis–Khan, 2010)

Consider all homeomorphisms $\sigma : T^n \to T^n$ such that:

- $\sigma^2 = \mathrm{id}$ *on* T^n, *and*
- $\sigma_* = -\mathrm{id}$ *on* $H_1(T^n; \mathbb{Z})$.

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1. *If* $n \equiv 0, 1 \pmod 4$ *or* $n = 3$, *then there is a unique conjugacy class of such involutions* σ, *represented by* $\sigma_0 : [x] \mapsto [-x]$.

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The proof follows from the next three technical theorems.

Equivariant Rigidity of the Torus, II

Before moving on, consider the following surgery-theoretic concept.

Definition

Let X be a topological manifold equipped with a cocompact action of a discrete group Γ. The **equivariant structure set** $\mathscr{S}_{\mathrm{TOP}}(X, \Gamma)$ is the set of equivalence classes of all pairs (M, f) such that:

- $M \subset \mathbb{R}^\infty$ is a manifold equipped with a cocompact Γ-action,
- $f : M \to X$ is a Γ-equivariant homotopy equivalence.

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Here, the pair (M,f) is equivalent to another such pair (M',f') if there exists a Γ-equivariant homeomorphism $h : M \to M'$ such that $f' \circ h$ is equivariantly homotopic to f.

Equivariant Rigidity of the Torus, III

Using Smith theory, we quantify those involutions to count them.

Theorem (Connolly–Davis–Khan, 2010)

The set of conjugacy classes of topological involutions

$$\sigma : T^n \to T^n \text{ such that } \sigma_* = -\mathrm{id} \text{ on } H_1(T^n)$$

is in bijective correspondence with the **equivariant structure set**

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Here, the action of C_2 on $T^n = S^1 \times \cdots \times S^1$ is generated by

$$\sigma_0 : (z_1, \ldots, z_n) \mapsto (\overline{z}_1, \ldots, \overline{z}_n).$$

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Suppose $n \geq 4$ and write $\varepsilon := (-1)^n$. Then

$$\mathscr{S}_{\mathrm{TOP}}(T^n, C_2) \ \cong \ \mathscr{S}_{\mathrm{TOP}}(\mathbb{R}^n, \Gamma_n) \ \cong \ \bigoplus \mathrm{UNil}_{n+\varepsilon}(D).$$

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Here, the direct sum is indexed by conjugacy classes (D) of maximal infinite dihedral subgroups $D \cong D_\infty$ of Γ_n.

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The above abelian groups have already been calculated.

Theorem (Banagl–Connolly–Davis–Ranicki, 2004)

For each integer m, there is an isomorphism

$$\mathrm{UNil}_m(D_\infty) \;\cong\; \begin{cases} 0 & \textit{if } m \equiv 0 \pmod 4 \\ 0 & \textit{if } m \equiv 1 \pmod 4 \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus\infty} & \textit{if } m \equiv 2 \pmod 4 \\ (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z})^{\oplus\infty} & \textit{if } m \equiv 3 \pmod 4. \end{cases}$$

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The last two answers are given by generalized Arf invariants.

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The Borel Conjecture has an equivariant generalization.

Conjecture (Quinn, 1986, modified from original)

Let Γ be a discrete group with no elements of order two. Let X and M be topological manifolds without boundary equipped with cocompact, proper actions of Γ such that:

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Let $f : M \to X$ be a **simple** *Γ-isovariant homotopy equivalence. Then f is isovariantly homotopic to a homeomorphism.*

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In this case, we say (X, Γ) is **isovariantly topologically rigid**.

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Let Γ be a discrete group. Let M and X be cocompact Γ-ANRs. Suppose $f : M \to X$ is a Γ-equivariant homotopy equivalence.

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Remark

If we drop the "simple" hypothesis in the previous conjecture, then counterexamples were constructed by Connolly–Koźniewski (1991).

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Let X be a topological space equipped with an action of group Γ. The action is **pseudofree** if the **singular set** X_{sing} is discrete:

$$X_{sing} \; := \; \{x \in X \mid \Gamma_x \neq 1\}.$$

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Theorem (Connolly–Khan, 2010)

Let X be a topological manifold of dimension $n \geq 5$ equipped with a $\mathrm{CAT}(0)$ metric. Let Γ be a discrete group of isometries of X.

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Theorem (Connolly–Khan, 2010)

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Theorem (Connolly–Khan, 2010)

Let X be a topological manifold of dimension $n \geq 5$ equipped with a $\mathrm{CAT}(0)$ metric. Let Γ be a discrete group of isometries of X. Suppose the action is cocompact, proper, and **pseudofree**. *Then*

$$\mathscr{S}_{\mathrm{TOP}}(X, \Gamma) \;\cong\; \bigoplus \mathrm{UNil}_{n+1}(D, \varepsilon).$$

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Corollary (Connolly–Khan, 2010)

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Consequently, we generalize the Bartels–Lück Theorem, which is the case when Γ is torsion-free.

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Consequently, we generalize the Bartels–Lück Theorem, which is the case when Γ is torsion-free. Our ideas further develop the topological application of their recent solution to the Farrell–Jones Conjecture. This is a powerful algebraic tool we discuss next time...