On the existence and classification of isovariant maps

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Today's talk

(I) Existence or nonexistence results on isovariant maps, in particular, Borsuk-Ulam type results.

(II) Classification results on isovariant maps, in particular, Hopf type results.

Isovariant maps

Let G be a compact Lie group. All maps are assumed to be continuous.

Definition. A G-map $f : X \to Y$ between G-spaces is called G-*i*sovariant if $G_{f(x)} = G_x$ for all $x \in X$.

Example.

(1) If both X and Y are free G-spaces, then an arbitrary G-map is a G-isovariant map.

(2) Suppose X is a G-space with nontrivial action and Y is a G-space with $Y^G \neq \emptyset$. In this case, a map $f : X \to Y^G \subset Y$ is equivariant, but not isovariant.

Isovariant homotopy classes

Definition. A G-homotopy $F : X \times I \to Y$ is called a *G-isovariant homotopy* if F is G-isovariant.

Let $[X, Y]_G^{\text{isov}}$ denote the set of isovariant homotopy classes of G-isovariant maps from X to Y.

As usual, $[X, Y]_G$ denotes the set of G-homotopy classes of G-maps from X to Y.

Notation

(1) C_n : a cyclic group of order n.

(2) U_k $(= \mathbb{C})$, $k \in \mathbb{Z}$: the unitary 1-dimensional representation of C_n on which a generator $c \in C_n$ acts by $c \cdot z = \xi_n^k z$, where $z \in U_k$ and $\xi_n = \exp(2\pi\sqrt{-1}/n)$.

(3) SV: the unit sphere of a representation V of G, which is called a representation sphere or linear G-sphere.

An existence problem

Let $G = C_{pq}$, where p, q are distinct primes.

Set

$$U_1^r = U_1 \oplus \cdots \oplus U_1 \ (r \text{ times})$$

and

$$W = U_p \oplus U_q.$$

Note that G acts freely on SU_1^r, but not freely on SW.

In fact, the singular set (nonfree part) of SW:

$$SW^{>1} = SW^{C_p} \bigcup SW^{C_q} = SU_p \coprod SU_q.$$

An existence problem

In equivariant case, as an application of equivariant obstruction theory, one can see that, for any $r \geq 1$, there exists a C_{pq}-map

$$g : SU_1^r \to S(U_p \oplus U_q).$$

Question. What about a C_{pq}-isovariant map?

Does there exist a C_{pq}-isovariant map from SU_1^r to SW?

The answer

If $r = 1$, then there is an isovariant map. For example, one can define an isovariant map $f_{0,0} : SU_1 \to SW$ by

$$f_{0,0}(z) = (z^p, z^q)/\sqrt{2}.$$

(In fact, $f_{0,0}$ is a G-embedding.)

When $r \geq 2$, the answer is "*No*."

This is shown by a Borsuk-Ulam type theorem.

Borsuk Ulam type theorems

In transformation group theory, the classical Borsuk-Ulam theorem is stated as follows.

Theorem 1. *Let S^m and S^n be spheres with antipodal C_2-action. If there is a C_2-map $f : S^m \to S^n$, then the inequality $m \leq n$ holds.*

Thus the Borsuk-Ulam theorem provides the nonexistence of a C_2-map. In fact, if $m > n$, then there is no C_2-map from S^m to S^n.

A generalization of the Borsuk-Ulam theorem

Many generalizations of the Borsuk-Ulam theorem are known. The following is one of them.

Theorem 2 (N-Hara-Kawakami-Ushitaki, Biasi-de Mattos)**.** *Let X be a free C_n-space and Y a Hausdorff free C_n-space. Suppose that there exists $m \geq 1$ such that*

$$\widetilde{H}_q(X;\mathbb{Z}/n) = 0 \ \ \textit{for} \ \ 0 \leq q \leq m, \quad \textit{and}$$

$$H_{m+1}(Y/C_n;\mathbb{Z}/n) = 0.$$

Then there is no C_n-map from X to Y.

Here the homology is the singular homology.

A generalization of the Borsuk-Ulam theorem

This theorem deduces a well-known result below.

Corollary 3 (mod p Borsuk-Ulam theorem)**.** *Assume that C_p (p: prime) acts freely on X with $H_*(X;\mathbb{Z}/p) \cong H_*(S^m;\mathbb{Z}/p)$ and on (Hausdorff) Y with $H_*(Y;\mathbb{Z}/p) \cong H_*(S^n;\mathbb{Z}/p)$. If there is a C_p-map $f : X \to Y$, then $m \leq n$.*

In other words, if $m > n$, then there is no C_p-map from X to Y.

Proof of the nonexistence

It suffices to show this when $r = 2$. Suppose

$$f : SU_1^2 = S(U_1 \oplus U_1) \to S(U_p \oplus U_q) = SW$$

is an isovariant map.

By restricting the action, we get a C_p-map $f : SU_1^2 \to SW \setminus SW^{C_p}$

Since $SW \setminus SW^{C_p} \simeq S^1$, $SW \setminus SW^{C_p}$ is a free C_p-homology sphere of (homological) dimension 1.

By the mod p Borsuk-Ulam theorem, we have $\dim SU_1^2 \leq 1$, however this is a contradiction. $\square$

Homologically linear actions

The above example is generalized.

Set

$$R_G = \begin{cases} \mathbb{Z}/|G| & \text{if } \dim G = 0, \\ \mathbb{Z} & \text{if } \dim G > 0. \end{cases}$$

Definition. A smooth closed G-manifolds Σ is called an *R_G-homologically linear G-sphere* if for every (closed) subgroup H, the H-fixed point set Σ^H is an R_G-homology sphere or the empty set; namely,

$$H_*(\Sigma^H; R_G) \cong H_*(S^{m(H)}; R_G), \ m(H) = \dim \Sigma^H.$$

For convenience, we set $\dim \Sigma^H = -1$ if Σ^H is empty.

More general results

Then we have

Theorem 4 (Isovariant Borsuk-Ulam theorem)**.** *Let G be a solvable compact Lie group. Let Σ_1 and Σ_2 be R_G-homologically linear G-spheres. If there is a G-isovariant map $f : \Sigma_1 \to \Sigma_2$, then the inequality*

$$\dim \Sigma_1 - \dim \Sigma_1^G \leq \dim \Sigma_2 - \dim \Sigma_2^G$$

holds.

Remark. Wasserman first proved this result for representation spheres.

Nonsolvable case

Using a result of Oliver, we have

Proposition 5. *If G is nonsolvable, then there exists a sequence*

$$\cdots \xrightarrow{h_n} \Sigma_n \xrightarrow{h_{n-1}} \Sigma_{n-1} \xrightarrow{h_{n-2}} \cdots \xrightarrow{h_1} \Sigma_1$$

such that

- *each Σ_n is a homologically linear G-sphere (in fact Σ_n can be taken to be a semilinear G-sphere),*
- *each h_n is a G-isovariant map,*
- $\Sigma_n^G = \emptyset$ *and* $\lim_{n\to\infty} \dim \Sigma_n = \infty.$

Nonsolvable case

Take a G-embedding $i : \Sigma_1 \subset SW$ for some representation W. Then an isovariant map $f_n : \Sigma_n \to SW$ is defined by composition.

Thus there is an integer n_0 such that

$$\dim \Sigma_n + 1 > \dim SW - \dim SW^G \text{ for any } n > n_0.$$

This shows that the isovariant Borsuk-Ulam theorem does not hold for a nonsolvable compact Lie group G.

Remark

Hence, for R_G-homologically linear actions, the isovariant Borsuk-Ulam theorem holds if and only if G is solvable.

Remark. The problem whether the above Σ_n can be taken to be a linear G-sphere is still open.

In equivariant case, the following is known.

Proposition 6 (Bartsch)**.** *Let G be a finite group. the Borsuk-Ulam theorem (in a weak sense) holds if and only if G is of prime power order.*

Corollary

Another result is obtained from the isovariant Borsuk-Ulam theorem.

Corollary 7 (N-Ushitaki)**.** *Let G be a finite group and Σ an R_G-homology sphere with free G-action. Let SW be the representation sphere of a representation W of G. If there is a G-isovariant map $f : \Sigma \to SW$, then the inequality*

$$\dim \Sigma + 1 \leq \dim SW - \dim SW^{>1},$$

where $SW^{>1} = \bigcup_{1 \neq H \leq G} SW^H$ (the singular set).

Remark. This result still holds when $G = S^1$, $\mathrm{Pin}(2) \cong N_{S^3}(S^1)$. However, in the case of $G = S^3$, it is unknown.

Existence results

The isovariant Borsuk-Ulam theorem provides nonexistence results of isovariant maps.

we here discuss an existence problem under some conditions.

As is seen before, there is a C_{pq}-isovariant map

$$f_{0,0} : SU_1 \to S(U_p \oplus U_q).$$

This is generalized as follows.

Existence results

Proposition 8. *Let M be a free G-manifold. Let W be a representation of G. Suppose*

$$\dim M/G + 1 \leq \dim SW - \dim SW^{>1}.$$

Then there exists a G-isovariant map from M to SW.

Outline of Proof.

Set $SW_{\mathrm{free}} = SW \setminus SW^{>1}$ and $d = \dim SW - \dim SW^{>1}$.

Fact. SW_{free} is $(d-2)$-connected.

Existence results

It suffices to construct a G-map from M to SW_{free}.

Fix a G-CW complex structure of M. One can inductively construct a G-map as follows.

Suppose that a G-map $f_k : X_k \to SW_{\mathrm{free}}$ is constructed on the k-skeleton X_k of M.

Let $X_{k+1} = X_k \cup_\phi G \times D^{k+1} \cup \cdots$. Then

$$f_k \circ \phi|\partial D^{k+1} : 1 \times \partial D^{k+1} \to SW_{\mathrm{free}}$$

is extended to $f_{k+1}| : D^{k+1} \to SW_{\mathrm{free}}$, since $k \leq d-2$ and SW_{free} is $(d-2)$-connected. Hence f_k equivariantly extends to a G-map $f_{k+1} : X_{k+1} \to SW_{\mathrm{free}}$. □

Classification problem — An example

Next we discuss a classification problem. Let $G = C_{pq}$. Recall the isovariant map

$$f_{0,0} : SU_1 \to SW = S(U_p \oplus U_q)$$

$$f_{0,0}(z) = (z^p, z^q)/\sqrt{2}.$$

One can find other isovariant maps. Indeed, a map $f_{\alpha,\beta} : SU_1 \to SW$ defined by

$$f_{\alpha,\beta}(z) = (z^{p(1+\alpha q)}, z^{q(1+\beta p)})/\sqrt{2},$$

$f_{\alpha,\beta}$ is G-isovariant for $(\alpha, \beta) \in \mathbb{Z}^2$.

Question. Do these maps represent different isovariant homotopy classes?

Classification problem — An example

The answer is "*Yes.*" In fact,

Proposition 9. *If $f_{\alpha,\beta}$ and $f_{\alpha',\beta'}$ are isovariantly homotopic, then $(\alpha, \beta) = (\alpha', \beta')$.*

In order to show this, we introduce the *multidegree* as an isovariant homotopy invariant.

If f is an isovariant map, then we obtain a G-map $f : SU_1 \to SW_{\mathrm{free}}$.

Consider the induced homomorphism

$$f_* : H_1(SU_1) \to H_1(SW_{\mathrm{free}}).$$

Classification problem — An example

Lemma 10.

$$\pi_1(SW_{\mathrm{free}}) \cong H_1(SW_{\mathrm{free}}) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Proof. $SW_{\mathrm{free}} = SW \setminus (SU_p \cup SU_q) \cong (U_p^{\perp} - 0) \times (U_q^{\perp} - 0) \simeq SU_q \times SU_p$. □

We define the multidegree $\mathrm{mDeg}\,(f)$ of f by

$$\mathrm{mDeg}\,(f) = f_*([SU_1]) \in \mathbb{Z} \oplus \mathbb{Z}.$$

Classification problem — An example

The multidegree of $f_{\alpha,\beta}$ is

$$\operatorname{mDeg} f_{\alpha,\beta} = (q(1+\beta p), p(1+\alpha q)).$$

This shows that if $(\alpha, \beta) \neq (\alpha', \beta')$, then $\operatorname{mDeg} f_{\alpha,\beta} \neq \operatorname{mDeg} f_{\alpha',\beta'}$.

Hence the isovariant maps $f_{\alpha,\beta}$ represent different isovariant homotopy classes.

Classification problem — An example

In this case, the converse is true; in fact,

Proposition 11. *Let* $f, g : SU_1 \longrightarrow SW$ *be isovariant maps. If* $\mathrm{mDeg}\, f = \mathrm{mDeg}\, g$*, then* f *and* g *are isovariantly homotopic.*

Outline of Proof.

Set $G = C_{pq}$.

It suffices to construct a G-homotopy $F : SU_1 \times I \longrightarrow SW_{\mathrm{free}}$ between f and g.

Classification problem — An example

Consider the commutative diagram:

$$
\begin{array}{ccc}
[SU_1, SW_{\mathrm{free}}]_G & \xrightarrow[\cong]{\gamma_G} & H^1(SU_1/G, \pi_1) = \mathbb{Z}^2 \\
\varepsilon \Big\downarrow & & \Big\downarrow \pi^* \\
[SU_1, SW_{\mathrm{free}}] & \xrightarrow[\gamma]{\cong} & H^1(SU_1, \pi_1) = \mathbb{Z}^2,
\end{array}
$$

where $\pi_1 = \pi_1(SW_{\mathrm{free}}) = \mathbb{Z}^2$. The vertical map ε is the forgetful map and $\pi : SU_1 \to SU_1/G$ is the orbit map.

Classification problem — An example

$$\begin{array}{ccc}
[SU_1, SW_{\mathrm{free}}]_G & \xrightarrow[\cong]{\gamma_G} & H^1(SU_1/G, \pi_1) = \mathbb{Z}^2 \\
\varepsilon \Big\downarrow & & \Big\downarrow \pi^* \\
[SU_1, SW_{\mathrm{free}}] & \xrightarrow[\gamma]{\cong} & H^1(SU_1, \pi_1) = \mathbb{Z}^2,
\end{array}$$

Fix a G-map $g : SU_1 \to SW_{\mathrm{free}}$. The horizontal maps are defined by

$$\gamma_G([f]) = \mathfrak{o}_G(f, g) \text{ and } \gamma([f]) = \mathfrak{o}(f, g),$$

which are bijections as a consequence of the equivariant obstruction theory.

Classification problem — An example

$$\begin{array}{ccc} [SU_1, SW_{\text{free}}]_G & \xrightarrow[\cong]{\gamma_G} & H^1(SU_1/G, \pi_1) = \mathbb{Z}^2 \\ \varepsilon \downarrow & & \downarrow \pi^* \\ [SU_1, SW_{\text{free}}] & \xrightarrow[\gamma]{\cong} & H^1(SU_1, \pi_1) = \mathbb{Z}^2, \end{array}$$

One can see that

π^* is multiplication by pq

and

$$\pi^*(\mathfrak{o}_G(f,g)) = \mathfrak{o}(f,g).$$

Classification problem — An example

$$\begin{array}{ccc} [SU_1, SW_{\text{free}}]_G & \xrightarrow[\cong]{\gamma_G} & H^1(SU_1/G, \pi_1) = \mathbb{Z}^2 \\ \varepsilon \downarrow & & \downarrow \pi^* \\ [SU_1, SW_{\text{free}}] & \xrightarrow[\gamma]{\cong} & H^1(SU_1, \pi_1) = \mathbb{Z}^2, \end{array}$$

Hence π^* is injective, and the forgetful map ε is injective. By calculation of the obstruction class, we have

$$\gamma([f]) = \mathfrak{o}(f, g) = \operatorname{mDeg} f - \operatorname{mDeg} g.$$

Hence if $\operatorname{mDeg} f = \operatorname{mDeg} g$, then we have $\mathfrak{o}_G(f, g) = 0$ and so a G-map $f \coprod g$ extends to a G-homotopy F. □

A classification result

Furthermore it is seen that

$$\mathrm{mDeg}\, f - \mathrm{mDeg}\, g \in pq\mathbb{Z}^2.$$

Taking $g = f_{0,0}$, we can define an injective map

$$D : [SU_1, SW_{\text{free}}]_G \to \mathbb{Z} \oplus \mathbb{Z}$$

by $D[f] = (\mathrm{mDeg}\, f - \mathrm{mDeg}\, f_{0,0})/pq$.

Since $D([f_{\alpha,\beta}]) = (\beta, \alpha)$, it follows that D is surjective. Hence D is a bijection.

A classification result

Thus we have the following classification result.

Proposition 12. *There is a one-to-one correspondence*

$$D : [SU_1, SW]_G^{\textit{isov}} \to \mathbb{Z} \oplus \mathbb{Z}.$$

In particular, the maps $f_{\alpha,\beta}$ represent all isovariant homotopy classes.

Using the notion of degree, H. Hopf showed that

$$\deg : [M, S^n] \to \mathbb{Z}$$

is a bijection for an orientable closed n-manifold M. We call this sort of result a *Hopf type theorem*.

A Hopf type theorem

The above example is generalized as follows.

We assume the following.

- G is a finite group.
- M is a connected, closed *free* G-manifold.
- SW is a *unitary* representation sphere of G.
- $\dim M + 1 = \dim SW - \dim SW^{>1}$.

Notation

- $\mathcal{A} = \{ H \in \operatorname{Iso} W \mid \dim SW^H = \dim SW^{>1} \}$.
- $\mathcal{A}/G = \{ (H) \mid H \in \mathcal{A} \}$.

A Hopf type theorem

Theorem 13 (Isovariant Hopf theorem)**.** *With the above assumption*

(1) *If M is orientable and the G-action on M is orientation-preserving, then there is a one-to-one correspondence*

$$[M, SW]_G^{\mathit{isov}} \cong \bigoplus_{(H)\in\mathcal{A}/G} \mathbb{Z}.$$

Every isovariant homotopy class is determined by the multidegree.

A Hopf type theorem

(2) *If M is non-orientable, then there is a one-to-one correspondence*

$$[M, SW]_G^{\textit{isov}} \cong \bigoplus_{(H) \in \mathcal{A}/G} \mathbb{Z}/2.$$

If G is of odd order, then every isovariant homotopy class is determined by the mod 2 multidegree.

Further results

(1) In the case where M is orientable, if the G-action is *not* orientation-preserving, then some $\mathbb{Z}/2$ components appear in $[M, SW]_G^{\mathrm{isov}}$, and the multidegree does not determine the isovariant homotopy classes.

$$[M, SW]_G^{\mathrm{isov}} \cong \bigoplus \mathbb{Z} \oplus \bigoplus \mathbb{Z}/2.$$

(2) In the case where M is non-orientable, if G is not of odd order, then the mod 2 multidegree does not always determine the isovariant homotopy classes.